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Lecture 1

Determinants

1: Definition of a Matrix:

If m and n are positive integers, an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

In which each entry , a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry in the ith row and jth column is denoted by the double subscripts notation a_{ij}. For instance a_{23} refers to the entry in the second row, third column. A matrix having m rows and n columns is said to be of <u>order</u> $m \times n$.

<u>Example 1:</u>

Determine the order of each matrix.

1: $[4]$ $\quad 2$: $[1 \quad 6 \quad -4]$ $\quad 3$: $\begin{bmatrix} 0 \\ 9 \end{bmatrix}$ $\quad 4$: $\begin{bmatrix} 7 & 3 \\ -1 & -7 \end{bmatrix}$ $\quad 5$: $\begin{bmatrix} 3 & 0 \\ 8 & 7 \\ 6 & 1 \end{bmatrix}$ $\quad 6$: $\begin{bmatrix} 4 & 0 & 5 \\ 3 & 2 & 1 \end{bmatrix}$

<u>Solution:</u>

1: The order of the matrix is 1×1 .

2: The order of the matrix is 1×3 .

3: The order of the matrix is 2×1 .

4: The order of the matrix is 2×2 .

5: The order of the matrix is 3×2 .

6: The order of the matrix is 2×3 .

2: Type of Matrices

1: Row Matrix:

A matrix having only one row is called *a row matrix*. For example:

$A = [3 \quad 6 \quad 9]$, $B = [0 \quad 1 \quad 5 \quad 7]$. **Here, the matrix A is of order 1×3, while the matrix B is of order 1×4.**

2: Column Matrix:

A matrix having only one *column is called a column matrix.*

For example: $A = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -2 \\ 8 \\ 0 \\ -5 \end{bmatrix}$. **Here, the matrix A *is of order* 2×1, while the matrix B is of the order 5×1.**

3: Square Matrix:

If the number of rows equal the number of columns, the matrix is called a square matrix. For example: $A = \begin{bmatrix} 4 & 0 \\ 7 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. **Here, the matrix A *is of order* 2×2, while the matrix B is of order 3×3.**

Note 1:

The square matrix has two diagonals are called the main diagonal which is from the top left corner to the bottom right corner and the other anti-diagonal which is from the top of right corner to the bottom of the left corner.

For example, for the square matrix A the main diagonal is $4, 3$.

4: Zero Matrix (Null Matrix):

A matrix whose elements are all zeros is called a zero matrix, and it is denoted *by O*. For example: $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, *or* $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

5: Identity Matrix (Unit Matrix):

A square matrix in which all elements of the main diagonal are equal to ones while all the others are zeros is called identity matrix or unit matrix,

It is $denoted\ by\ I$. For example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6: Diagonal Matrix:

A square matrix for which $a_{jk} = 0\ when\ j \neq k$ is called a diagonal matrix. For example:

$$A = \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix}\ ,\ B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

7: Transpose of a Matrix:

If we interchange rows and columns of a matrix A, then the resulting matrix is called the transpose of $the\ matrix$ A, and is $denoted\ by\ A^T$.

$$For\ example\ if\ A = \begin{bmatrix} 8 & 6 \\ 4 & 3 \\ 2 & 1 \end{bmatrix}, then\ A^T = \begin{bmatrix} 8 & 4 & 2 \\ 6 & 3 & 1 \end{bmatrix}.$$

8: Symmetric Matric:

$A\ square\ matrix\ A\ is\ called\ symmetric\ if\ A^T = A$. **For example:**

The matrix $A = \begin{bmatrix} 4 & 5 & 8 \\ 5 & 3 & 2 \\ 8 & 2 & 1 \end{bmatrix}$ **is symmetric because** $A^T = \begin{bmatrix} 4 & 5 & 8 \\ 5 & 3 & 2 \\ 8 & 2 & 1 \end{bmatrix} = A$.

9: Triangular Matrix:

(i) ***Lower Triangular Matrix***:

A square matrix A is called a lower triangular matrix if $a_{ij} = 0\ for\ i < j$.

That is: all elements above the main diagonal are zeros. For example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 6 & 4 & 0 \\ 1 & 3 & 5 \end{bmatrix},\ \begin{bmatrix} 7 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 9 & 6 & 3 & 1 \end{bmatrix}.$$

(ii) ***Uper Trangular Matrix***:

A square matrix A is called an upper triangular $matrix\ if\ a_{ij} = 0, for\ i > j$. That is: all elements below the main diagonal are zeros. For example:

$$A = \begin{bmatrix} 5 & 7 & 1 \\ 0 & 8 & 3 \\ 0 & 0 & 6 \end{bmatrix},\ B = \begin{bmatrix} 3 & 9 & 4 & 2 \\ 0 & 8 & 7 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

2: Definition of Determinant:

If A is a square matrix, we associate with A a real number denoted by $det(A)$ or $|A|$ called the determinant of the matrix A.

We can calculate the determinant of a square matrix as follows:

1: If the matrix A is of order 1×1

This matrix has only one element, say $a. Then\ det(A) = a$.

For example:

If $A = [6]\ , then\ det(A) = 6$.

If $B = [-3]\ , then\ det(B) = -3$.

2: If the matrix A is of order 2×2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, then\ det(A) = ad - bc$.

Example 2:

Find the determinant of the following matrices:

$(i)\ A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}\quad (ii)\ B = \begin{bmatrix} 2 & 5 \\ 4 & 10 \end{bmatrix}\quad (iii)\ C = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$

Solution:

$(i)\ det(A) = (4)(3) - (-2)(1) = 12 + 2 = 14.$

$(ii)\ det(B) = (2)(10)_{(5)(4)} = 20 - 20 = 0.$

$(iii)\ det(C) = (0)(2) - (1)(3) = 0 - 3 = -3.$

3: If the matrix A is of order 3×3

$$Let\ the\ matrix\ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$ **Then**

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$= [(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}]$$

$$-[a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33}]$$

Example 3:

Find the determinant of each matrix:

$$(i)\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}\quad (ii)\ B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix}\quad (iii)\ C = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 8 \end{bmatrix}.$$

Solution:

$$(i)\ det(A) = |A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ 2 & 1 \\ 3 & 2 \end{matrix}$$

$$= [(1)(1)(1) + (2)(3)(3) + (3)(2)(2)]$$

$$-[(3)(1)(3) + (1)(3)(2) + (2)(2)(1)]$$

$$= [1 + 18 + 12] - [9 + 6 + 4] = 31 - 19 = 12.$$

$$(ii)\ det(B) = |B| = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 5 \\ 0 & 0 \end{matrix}$$

$$= [(1)(5)(-6) + (0)(0)(0) + (2)(0)(0)]$$

$$-[(2)(5)(0) + (1)(0)(0) + (0)(0)(-6)]$$

$$= [-30 + 0 + 0] - [0 + 0 + 0] = -30 - 0 = -30.$$

$$(iii)\ det(C) = |C| = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 5 & 7 & 8 \end{bmatrix} \begin{matrix} 2 & 4 \\ 1 & 2 \\ 5 & 7 \end{matrix}$$

$$= [(2)(2)(8) + (4)(3)(5) + (6)(1)(7)]$$

$$-[(6)(2)(5) + (2)(3)(7) + (4)(1)(8)]$$

$$= [32 + 60 + 42] - [60 + 42 + 32] = 134 - 134 = 0.$$

3: The General Definition of Determinant:

To define the determinant of a square matrix of order 3×3 or higher, it is convenient to introduce the concepts of minors and cofactors.

If A is a square matrix, the minor m_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the $ith\ row\ and\ jth\ column\ of\ A$.

The cofactor of the element a_{ij} is $c_{ij} = (-1)^{i+j} m_{ij}$.

Example 4:

$$Find\ all\ minors\ and\ cofactors\ for\ the\ matrix\ A = \begin{bmatrix} 0 & 2 & -3 \\ 3 & -1 & -2 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution:

The minors are

$$m_{11} = \begin{vmatrix} -1 & -2 \\ 5 & 6 \end{vmatrix} = (-1)(6) - (-2)(5) = -6 + 10 = 4.$$

$$m_{12} = \begin{vmatrix} 3 & -2 \\ 4 & 6 \end{vmatrix} = (3)(6) - (-2)(4) = 18 + 8 = 26.$$

$$m_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} = (3)(5) - (-1)(4) = 15 + 4 = 19.$$

$$m_{21} = \begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix} = (2)(6) - (-3)(5) = 12 + 15 = 27.$$

$$m_{22} = \begin{vmatrix} 0 & -3 \\ 4 & 6 \end{vmatrix} = (0)(6) - (-3)(4) = 0 + 12 = 12.$$

$$m_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} = (0)(5) - (2)(4) = 0 - 8 = -8 .$$

$$m_{31} = \begin{vmatrix} 2 & -3 \\ -1 & -2 \end{vmatrix} = (2)(-2) - (-3)(-1) = -4 - 3 = -7 .$$

$$m_{32} = \begin{vmatrix} 0 & -3 \\ 3 & -2 \end{vmatrix} = (0)(-2) - (-3)(3) = 0 + 9 = 9 .$$

$$m_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = (0)(-1) - (2)(3) = 0 - 6 = -6 .$$

The cofactors are

$c_{11} = (-1)^{1+1} m_{11} = m_{11} = 4 .$

$c_{12} = (-1)^{1+2} m_{12} = -m_{12} = -26 .$

$c_{13} = (-1)^{1+3} m_{13} = m_{13} = 19 .$

$c_{21} = (-1)^{2+1} m_{21} = -m_{21} = -27 .$

$c_{22} = (-1)^{2+2} m_{22} = m_{22} = 12 .$

$c_{23} = (-1)^{2+3} m_{23} = -m_{23} = 8 .$

$c_{31} = (-1)^{3+1} m_{31} = m_{31} = -7 .$

$c_{32} = (-1)^{3+2} m_{32} = -m_{32} = -9 .$

$c_{33} = (-1)^{3+3} m_{33} = m_{33} = -6 .$

Now, we give the general definition of determinant as follows:

If A is a square matrix (of order 3×3 $or\ greater$), <u>the determinant of the matrix A is the sum of the elements in any row (or column) of A multiplied by their respective cofactors.</u> That is:

$$det(A) = |A| = \sum_{k=1}^{n} a_{jk} c_{jk} .$$

<u>Example 5:</u>

Use the general definition to find the determinant of the $A = \begin{bmatrix} 0 & 2 & -3 \\ 3 & -1 & -2 \\ 4 & 5 & 6 \end{bmatrix}$.

<u>Solution:</u>

Take the first row.

$$det(A) = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 0 + 2(-26) + (-3)(19)$$
$$= 0 - 52 - 57 = -109\,.$$

Or take second row.

$$det(A) = a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} = 3(-27) + (-1)(12) + (-2)(8)$$
$$= -81 - 12 - 16 = -109.$$

Or take third row.

$$det(A) = a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33} = 4(-7) + 5(-9) + 6(-6)$$
$$= -28 - 45 - 36 = -109.$$

Or take first column.

$$det(A) = a_{11}c_{11} + a_{21}c_{21} + a_{31}c_{31} = 0 + 3(-27) + 4(-7)$$
$$= 0 - 81 - 28 = -109.$$

Or take second column.

$$det(A) = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} = 2(-26) + (-1)(12) + 5(-9)$$
$$= -52 - 12 - 45 = -109.$$

Or take third column.

$$det(A) = a_{13}c_{13} + a_{23}c_{23} + a_{33}c_{33} = -3(19) + (-2)(8) + 6(-6)$$
$$= -57 - 16 - 36 = -109.$$

<u>Example 6:</u>

$$Find\ the\ determinant\ of\ the\ matrix\ A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

<u>Solution:</u>

Take the first row.

$$det(A) = |A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix}$$

$$-2(3 - 8) + (0 + 4) = 10 + 4 = 14\,.$$

Or take second row.

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = -3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix}}$$

$$\boldsymbol{= -3(2-0) - (0-4) - 2(0-8)}$$

$$\boldsymbol{-6 + 4 + 16 = 14\,.}$$

Or take third row.

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = 4\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix}}$$

$$\boldsymbol{= 4(4+1) + (0-6) = 20 - 6 = 14\,.}$$

Or take first column.

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = -3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}}$$

$$\boldsymbol{= -3(2-0) + 4(4+1) = -6 + 20 = 14\,.}$$

Or take second column.

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix}}$$

$$\boldsymbol{= -2(3-8) - (0-4) = 10 + 4 = 14\,.}$$

Or take third column.

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix}}$$

$$\boldsymbol{= (0+4) - 2(0-8) + (0-6)}$$

$$\boldsymbol{= 4 + 16 - 6 = 14\,.}$$

<u>Note 2:</u>

If we use the general definition to evaluate the determinant of a square matrix, we choose always the row or the column contains zeros.

Example 7:

Use general definition to evaluate $\boldsymbol{the\ determinant\ of\ A} = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 4 & 5 \\ 0 & 0 & 8 \end{vmatrix}$.

Solution:

We can take third row or second column.

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 4 & 5 \\ 0 & 0 & 8 \end{vmatrix} = 8\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8(8-0) = 64}\ .$$

Or

$$\boldsymbol{det(A) = |A| = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 4 & 5 \\ 0 & 0 & 8 \end{vmatrix} = 4\begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} = 4(16-0) = 64}\ .$$

4: Some Properties of Determinants:

1: If all elements of any row (or column) of a square matrix are zeros, then its determinant equals to zero. For example:

Let $\boldsymbol{A = \begin{bmatrix} 0 & 0 \\ 3 & 5 \end{bmatrix}, then\ det(A) = 0 - 0 = 0}$.

Let $\boldsymbol{B = \begin{bmatrix} 3 & 0 \\ 7 & 0 \end{bmatrix}, then\ det(B) = 0 - 0 = 0}$.

2: If any two rows (or columns) of a square matrix are the same, then its determinant equals to zero. For example:

Let $\boldsymbol{A = \begin{bmatrix} 2 & 7 & 6 \\ 1 & 3 & 5 \\ 2 & 7 & 6 \end{bmatrix}, then}$

$$\boldsymbol{det(A) = 2\begin{vmatrix} 3 & 5 \\ 7 & 6 \end{vmatrix} - 7\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}}$$

$$\boldsymbol{= 2(18-35) - 7(6-10) + 6(7-6)}$$

$$\boldsymbol{= -34 + 28 + 6 = -34 + 34 = 0}.$$

3: If any two rows (or columns) of a square matrix are proportional, then its determinants equals to zero. For example $in\ the\ matrix\ A = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$**, we** $notice\ that\ \frac{2}{6} = \frac{4}{12}$**, this means first and second rows are proportional. In this case** $det(A) = 24 - 24 = 0$.

Also, we notice that $\frac{2}{4} = \frac{6}{12}$**, this means the first and second columns are proportional. In this case** $det(A) = 24 - 24 = 0$.

4: An interchange of any two rows (or columns) of a square matrix change the sign of its determinant.

For example: Let $A = \begin{bmatrix} 4 & 3 \\ -2 & 5 \end{bmatrix}$ $, then\ det(A) = 20 + 6 = 26.\ Let$ **the matrix**

$B = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix}, then\ det(B) = -6 - 20 = -26$.

5: The determinant of a square matrix equals to the determinant of its transpose. For example:

Let $A = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix}$ $, then\ det(A) = -6 + 4 = -2$.

Then $A^T = \begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix}, then\ det(A^T) = -6 + 4 = -2$.

$Therefore, det(A) = det(A^T)$.

6: If all elements in any row (or column) in a square matrix are multiplied by a number, then its determinant is also multiplied by the same number.

For example: Let $A = \begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix}, then\ det(A) = -6 + 4 = -2$.

We form a new matrix B **by multiplied the first column by a number 5. That is** $B = \begin{bmatrix} 15 & 1 \\ -20 & -2 \end{bmatrix}, then\ det(B) = -30 + 20 = -10 = 5(-2) = (5)det(A)$.

7: If we express the elements of each row (or column) of a square matrix as sum of two terms. Then the determinant of a matrix can be expressed as sum of two determinants. For example:

Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}, then\ det(A) = 12 - 5 = 7$**. We can write** $det(A)$ **as**

$$det(A) = \begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2+1 & 3+2 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = (8-3) + (4-2)$$

$$5 + 2 = 7\ .$$

Note 3:

The determinant of the triangular matrix equals the product of all elements on the main diagonal. For example:

Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & -1 & -2 \end{bmatrix}, det(A) = (4)(2)(-2) = -16$.

Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, det(B) = (1)(4)(6) = 24$.

Note 4:

The determinant of the diagonal matrix equals the product of all elements on the main diagonal. For example:

Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, then\ det(A) = (3)(2)(4) = 24$.

Note 5:

The determinant of the identity matrix equals 1.

Note 6:

If the matrix A **of order** $n \times n$ **with** $determinant\ equals\ \Delta$**. Let** $B = kA, k$ **is any number. Then** $det(B) = k^n \Delta$**. For xample:**

Let $A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}\ and\ k = 3. Then\ B = 3A = 3\begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 9 & -15 \end{bmatrix}$.

Now, $det(A) = -5 + 6 = 1\ \ and\ det(B) = -45 + 54 = 9 = 3^2 det(A)$.

5: Singular and Non-Singular Matrices:

A square matrix with determinant equals zero is called a singular matrix.

$$For\ example\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 9 & 7 \end{bmatrix}.$$

A square matrix is called a non-singular if its determinant is not equal zero.

H.W. 1

1: Find the determinant of each matrix.

$(i)\ A = [-8]\quad (ii)\ B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\quad (iii)\ C = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

$(iv)\ D = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}\quad (v)\ E = \begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$

$(vi)\ F = \begin{bmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{bmatrix}\quad (vii)\ G = \begin{bmatrix} 10 & -5 & 5 \\ 30 & 0 & 10 \\ 0 & 10 & 1 \end{bmatrix}.$

2: Find all minors and cofactors of each matrix.

$(i)\ A = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}\quad (ii)\ B = \begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}\quad (iii)\ C = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & -6 & 4 \end{bmatrix}.$

3: Solve for x .

$(i)\ \begin{vmatrix} x-1 & 2 \\ 3 & x-2 \end{vmatrix} = 0\quad (ii)\ \begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix} = 0\quad (iii)\ \begin{vmatrix} x+4 & -2 \\ 7 & x-5 \end{vmatrix} = 0$

$(iv)\ \begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0\quad (v)\ \begin{vmatrix} 2-x & 5 \\ 3 & -8-x \end{vmatrix} = 0\ .$

4: Find the value of x such that the matrix $A = \begin{bmatrix} 4 & x \\ -2 & -3 \end{bmatrix}$ is singular.

6: Cramer′s Rule:

First: For 2 × 2 System:

Let A be the coefficient of the following linear system

$$ax + by = s$$

$$cx + dy = t$$

Where a, b, c, d, s and t are $real$ $numbers$ and the $unknown$ are x and y. Let the $matrix$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $det(A) \neq 0$, then the above linear system has exactly one solution. The solution is given by:

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{det(A)} \text{ and } y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{det(A)} .$$

Example 8:

Use Cramer′s rule to solve the following linear system.

$$8x + 5y = 2$$

$$2x - 4y = -10$$

Solution:

1: $Write$ the $matrix$ of $coefficients$ of the $system$ as: $A = \begin{bmatrix} 8 & 5 \\ 2 & -4 \end{bmatrix}$.

2; Evaluate determinant of the matrix A.

$$det(A) = (8)(-4) - (5)(2) = -32 - 10 = -42 \neq 0 .$$

3: Find the values of x and y as $follows$:

$$x = \frac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{det(A)} = \frac{(2)(-4)-(5)(-10)}{-42} = \frac{-8+50}{-42} = \frac{42}{-42} = -1 .$$

And

$$y = \frac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{det(A)} = \frac{(8)(-10)-(2)(2)}{-42} = \frac{-80-4}{-42} = \frac{-84}{-42} = 2 .$$

Then the solution of the system is $(-1, 2)$.

Second: For 3×3 System:

Let A be the coefficient matrix of the following linear system:

$$ax + by + cz = j$$

$$dx + ey + fz = k$$

$$gx + hy + iz = l$$

Where $a, b, c, d, e, f, g, h, i, j, k$ and l are the $real$ $numbers$ and the $unknown$ are x, y and z.

$$The\ matrix\ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

If $det(A) \neq 0$, then the system has exactly one solution. The solution is given by:

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{det(A)}\ ,\ y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{det(A)}\ , and\ z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{det(A)}\ .$$

Example 9:

Use Cramer′s rule to solve the following linear system.

$$x + 2y - 3z = -2$$

$$x - y + z = -1$$

$$3x + 4y - 4z = 4$$

Solution:

1: Write the matrix of the coefficients of the system as $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 1 \\ 3 & 4 & -4 \end{bmatrix}$.

2: Evaluate the determinant of the matrix A.

$$det(A) = \begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= (-1)(-4) - (1)(4) - 2[(1)(-4) - (1)(3)] - 3[(1)(4) - (-1)(3)]$$

$$= 4 - 4 - 2[-4 - 3] - 3[4 + 3] = 0 + 14 - 21 = -7 \neq 0.$$

3: Evaluate the values of $x, y, and\ z$ as follows:

$$x = \frac{\begin{vmatrix} -2 & 2 & -3 \\ -1 & -1 & 1 \\ 4 & 4 & -4 \end{vmatrix}}{det(A)} = \frac{-2\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix}}{-7}$$

$$= \frac{-2[4-4]-2[4-4]-3[-4+4]}{-7} = \frac{0-0-0}{-7} = \frac{0}{-7} = 0 .$$

$$y = \frac{\begin{vmatrix} 1 & -2 & -3 \\ 1 & -1 & 1 \\ 3 & 4 & -4 \end{vmatrix}}{det(A)} = \frac{\begin{vmatrix} -1 & 1 \\ 4 & -4 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix}}{-7} = \frac{4-4+2(-4-3)-3(4+3)}{-7}$$

$$= \frac{0-14-21}{-7} = \frac{-35}{-7} = 5 .$$

$$z = \frac{\begin{vmatrix} 1 & 2 & -2 \\ 1 & -1 & -1 \\ 3 & 4 & 4 \end{vmatrix}}{det(A)} = \frac{\begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix}}{-7} = \frac{0-2(4+3)-2(4+3)}{-7} = \frac{-14-14}{-7}$$

$$= \frac{-28}{-7} = 4 .$$

Then the solution of the system is $(0, 5, 4)$.

<u>Example 10:</u>

Use Cramer′s rule to solve the following linear system.

$$x + 3y - z = 1$$

$$-2x - 6y + z = -3$$

$$3x + 5y - 2z = 4$$

<u>Solution:</u>

1: Write the matrix of the coefficients of the system as $A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & -6 & 1 \\ 3 & 5 & -2 \end{bmatrix}$.

2: Evaluate the determinant of the matrix A as follows.

$$det(A) = \begin{vmatrix} -6 & 1 \\ 5 & -2 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} - \begin{vmatrix} -2 & -6 \\ 3 & 5 \end{vmatrix}$$

$$= (12 - 5) - 3(4 - 3) - (-10 + 18) = 7 - 3 - 8 = -4 \neq 0 .$$

3: Evaluate $x, y, and\ z\ as\ follows$:

$$x=\frac{\begin{vmatrix}1&3&-1\\-3&-6&1\\4&5&-2\end{vmatrix}}{det(A)}=\frac{\begin{vmatrix}-6&1\\5&-2\end{vmatrix}-3\begin{vmatrix}-3&1\\4&-2\end{vmatrix}-\begin{vmatrix}-3&-6\\4&5\end{vmatrix}}{-4}=\frac{12-5-3(6-4)-(-15+24)}{-4}$$

$$=\frac{7-6-9}{-4}=\frac{-8}{-4}=2\ .$$

$$y=\frac{\begin{vmatrix}1&1&-1\\-2&-3&1\\3&4&-2\end{vmatrix}}{-4}=\frac{\begin{vmatrix}-3&1\\4&-2\end{vmatrix}-\begin{vmatrix}-2&1\\3&-2\end{vmatrix}-\begin{vmatrix}-2&-3\\3&4\end{vmatrix}}{-4}=\frac{(6-4)-(4-3)-(-8+9)}{-4}$$

$$=\frac{2-1-1}{-4}=\frac{2-2}{-4}=\frac{0}{-4}=0\ .$$

$$z=\frac{\begin{vmatrix}1&3&1\\-2&-6&-3\\3&5&4\end{vmatrix}}{det(A)}=\frac{\begin{vmatrix}-6&-3\\5&4\end{vmatrix}-3\begin{vmatrix}-2&-3\\3&4\end{vmatrix}+\begin{vmatrix}-2&-6\\3&5\end{vmatrix}}{-4}=\frac{(-24+15)-3(-8+9)+(-10+18)}{-4}$$

$$=\frac{-9-3+8}{-4}=\frac{-4}{-4}=1\ .$$

Then the solution of the system is $(2, 0, 1)$.

H.W. 2

Use Cramer′s rule to solve the following linear systems.

$\mathbf{1}$: $-8x+y=-6$, $-5x+4y=3$

2: $3x-2y=10$, $-6x+y=-7$

3: $5x+4y=12$, $3x-6y=3$

4: $4x+y+z=2$, $2x+2y+4z=1$, $-x-y+z=5$

5: $x+y+4z=7$, $2x-3y-z=-24$, $-4x+2y+2z=8$

6: $3x+3y-2z=-18$, $-5x-2y-3z=-1$, $7x+y+6z=14$

7: $3x=2y+3+z$, $2x-y+4=z$, $y+z=-x+3$

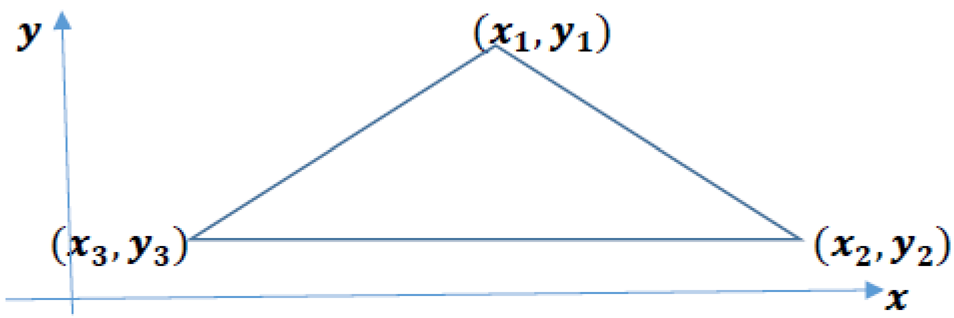
8: $3x+y-z=0$, $x+y+z=0$, $y-z=-1$

9: $x-z=3$, $y-z=1$, $2x+z=3$

10: $2x+y-z=2$, $x-y+z=7$, $2x+2y+z=4$

7: Some Applications of Determinant:

1: Area of a Triangle:



The area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) is given by

$$Area = \mp\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Where the sign $(\mp)$ is chosen to give a positive area.

Example 11:

Find the area of the triangle whose vertices are $(1, 0), (2, 2), and\ (4, 3)$.

Solution:

1: Evaluate the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} = (2-3) + (6-8)$$

$$= -1 - 2 = -3\ .$$

2: Evaluate the area of triangle as

$$Area = -\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = -\frac{1}{2}(-3) = \frac{3}{2}\ \ square\ units.$$

Example 12:

Find the area of the triangle whose vertices are $(1, -9), (2, 3), and\ (5, -7)$.

Solution:

1: Evaluate the determinant $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -9 & 1 \\ 2 & 3 & 1 \\ 5 & -7 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 1 \\ -7 & 1 \end{vmatrix} + 9\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & -7 \end{vmatrix}$$

$$= (3+7) + 9(2-5) + (-14-15)$$

$$= 10 - 27 - 29 = -46\,.$$

2: Evaluate the area of triangle as

$$Area = -\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = -\frac{1}{2}(-46) = 23 \;\; square\ units.$$

2: Show that Three Points Are Collinear:

Three points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) **are collinear if and only if**

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0\,.$$

Example 13:

Show that the points $(0,1), (2,2) and\ (4,3)\ are\ collinear.$

Solution:

Three points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) **are collinear if and only if**

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0\,.$$

Then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix}$

$$= -(2-4) + (6-8) = 2 - 2 = 0\,.$$

Then the points $(0,1), (2,2) and\ (4,3)\ are\ collinear.$

3: An Equation of a Line Passing Through Two Points:

An equation of the line passing through two distinct points (x_1, y_1) and (x_2, y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.

Example 14:

Find an equation of the line passing through the points $(2, 4)$ and $(-1, 3)$.

Solution:

$$\textit{The equation is given by } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 .$$

Then $\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \ \rightarrow \ x\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 0 . \rightarrow$

$$\rightarrow x(4-3) - y(2+1) + (6+4) = 0 \ \rightarrow$$

$$\rightarrow x - 3y + 10 = 0 .$$

H.W. 3

1: Find the area of the following triangles whose vertices have the given points

(i) $(-4, -1), (3, 2), (4, 6)$ (ii) $(4, -5), (3, -8)$ (iii) $(0, 0), (3, 0), (0, 2)$

(iv) $(-2, -2), (3, 4), (3, -2)$ (v) $(1, 0), (6, 5), (0, 2)$.

2: Determine whether the given points are collinear.

(i) $(0, -4), (2, 0), (3, 2)$ (ii) $(0, 4), (7, -6), (-5, 11)$

(iii) $(-1, -3), (-4, -9), (2, 3)$.

3: Find an equation of the line passing through each two points.

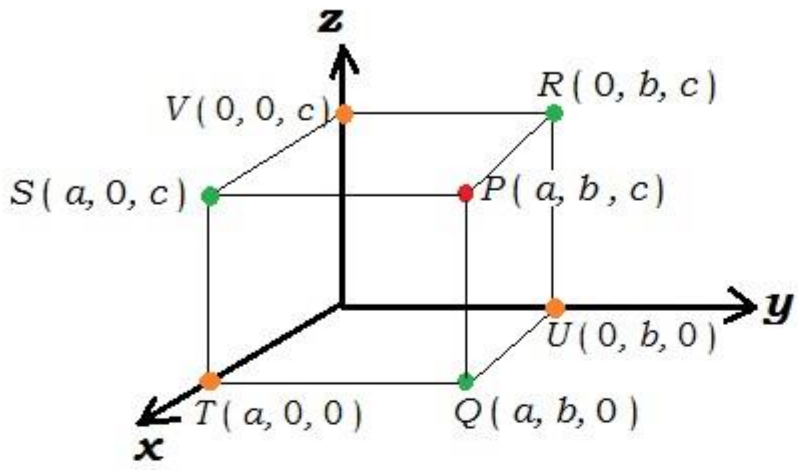
(i) $(-6, -1)$, $(-5, 6)$ (ii) $\left(-\frac{3}{4}, -\frac{1}{4}\right), \left(\frac{2}{7}, -\frac{5}{7}\right)$ (iii) $\left(\sqrt{2}, -4\right), \left(-\sqrt{2}, 2\right)$.

Lecture 2

Vectors in Space

1: Three Dimensional Coordinate Systems

To locate a point in space, we use three mutually perpendicular coordinate axes, arranged as in Figure.



The Cartesian coordinates (a, b, c) of a point P in space are the values at which the planes through P perpendicular to the axes cut the axes. Cartesian coordinates for space are also called rectangular coordinates because that define them meet at right angles. Points on the $x - axis$ have coordinates of the form $(a, 0, 0)$. Points on the $y - axis$ have coordinates of the form $0, b, 0)$. Points on the $z - axis$ have coordinates of the form $(0, 0, c)$.

The planes determined by the coordinate axes are:

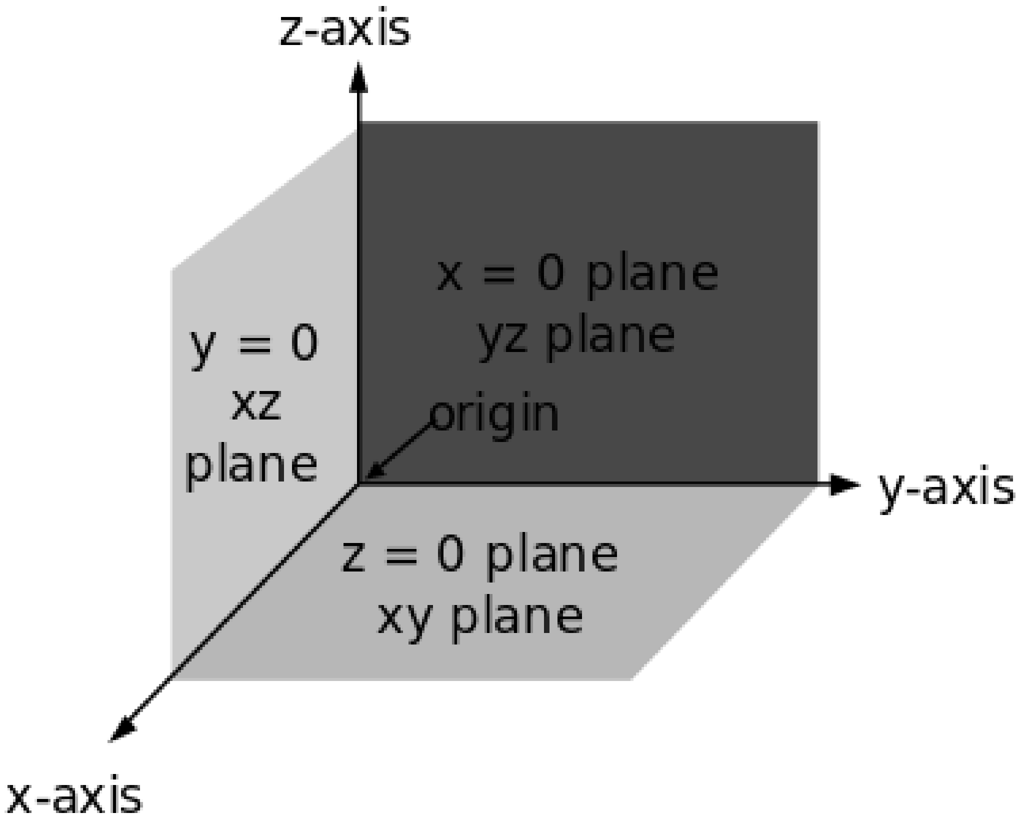
A: $xy - plane$, whose standard equation $z = 0$.

B: $yz - plane$, whose standard equation $x = 0$.

C: $xz - plane$, whose standard equation $y = 0$.

The planes meet at the origin $(0, 0, 0)$.The origin is also identified by simply 0 or sometimes the letter $O. The$ three coordinate planes $x = 0, y = 0$ & $z = 0$ $divide\ space$ into eight cells called octants. The octant in which the point coordinates are all positive is called the first octant. The points in a plane perpendicular to the $x - axis$ all $have\ same\ x - coordinate$. The points in a plane perpendicular to the $y - axis$ all have the same $y - coordinate$. The points in a plane perpendicular to the $z - axis$ all have the same $z - coordinate$. For example, the plane $x = 2$ is the plane perpendicular to the

$x - axis\ at\ x = 2$. The plane $y = 3$ is the plane perpendicular to the $y - axis\ at\ y = 3$. The plane $z = 5$ is the plane perpendicular to the $z - axis\ at\ z = 5$



The planes $x = 2$ & $y = 3$ intersect in a line parallel to the $z - axis$.
This line is described by the pair of equations $x = 2, y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ & $y = 3$.
Similarly, the line of intersection of the planes $y = 3$ & $z = 5$ is described by the pair $of\ equation$ $y = 3$, $z = 5$. This line runs parallel to the $x - axis$.
The line of the intersection of the planes $x = 2$ & $z = 5$ $parallel\ to\ the\ y - axis$, is described by the equation pair $x = 2$, $z = 5$.

Example (1):

Interpret the following equations and inequalities geometrically.

No.	Equation or Inequality	Interpret
1	$z \geq 0$	The half space consisting of the points on and above the $xy - plane$.

2	$x = -3$	The plane perpendicular to the $x - axis$ at $x = -3$.
3	$z = 0, x \leq 0, y \geq 0$	The second quadrant of the $xy - plane$.
4	$x \geq 0, y \geq 0, z \geq 0$	The first octant.
5	$-1 \leq y \leq 1$	The slab between the planes $y = -1$ and $y = 1$.
6	$y = -2, z = 2$	The line in which the planes $y = -2$ & $z = 2$ $intersect$. Alternatively, the line through the point $(0, -2, 2)$ parallel to the $x - axis$.

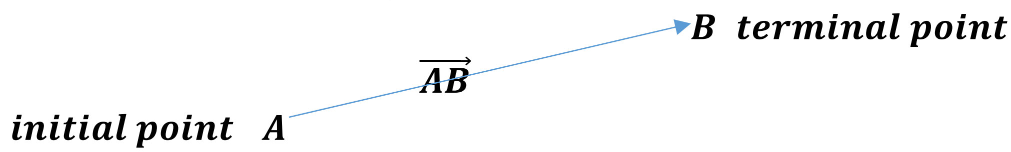
2: Vectors:

Definition (1):

A quantity determine by their magnitudes as well as its direction is called vector quantity, and it is represented by a direct line segment.

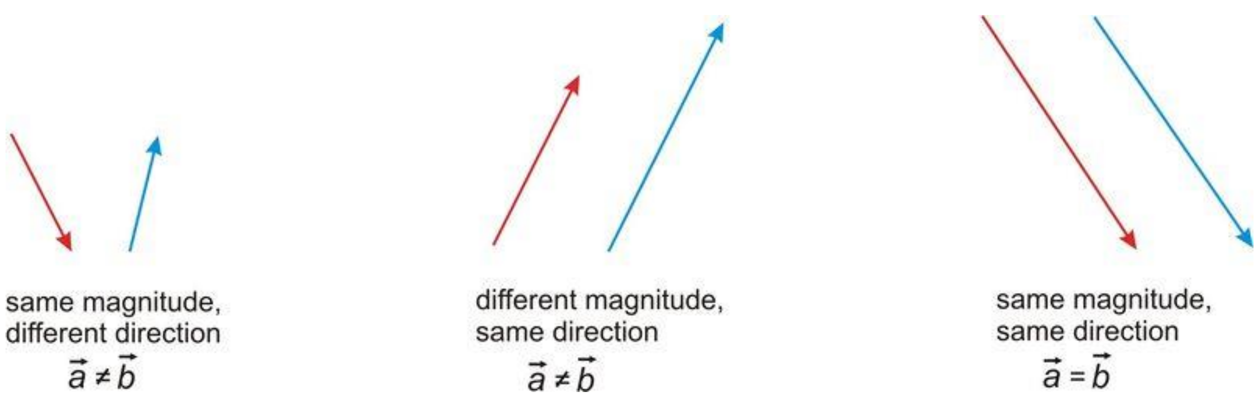
Definition (2):

The vector represented by the direct line segment $\overrightarrow{AB}$ has initial point A and terminal point B and its length $|\overrightarrow{AB}|$.



Definition (3):

Two vectors are equal if they have the same length and same direction.



Definition (4):

If v is a two dimensional vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the component form of v is $v = \langle v_1, v_2 \rangle$.

If v is a three dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the component form of v is $v = \langle v_1, v_2, v_3 \rangle$.

Notes 1:

1: Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ the standard position vector $v = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is:

$$v = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

2: Two vectors are equal if and only if their standard position vectors are identical. Thus

$$\langle u_1, u_2, u_3 \rangle = \langle v_1, v_2, v_3 \rangle \; if\ and\ only\ if\ u_1 = v_1, u_2 = v_2, u_3 = v_3.$$

3: The magnitude of the vector $v = \overrightarrow{PQ}$ is the nonnegative number

$$|v| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

4: The only vector with length zero is the zero vector $O = (0, 0, 0)$. $This\ vector\ is\ also\ the\ only\ vector\ with\ no\ specific\ direction.$

Example (2):

Find the component form and length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution:

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \langle -2, -2, 1 \rangle .$$

The components of $\overrightarrow{PQ}$ are:

$v_1 = -2, v_2 = -2 \;\&\; v_3 = 1$.

The length of $\overrightarrow{PQ} = |\overrightarrow{PQ}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = 3\ units.$

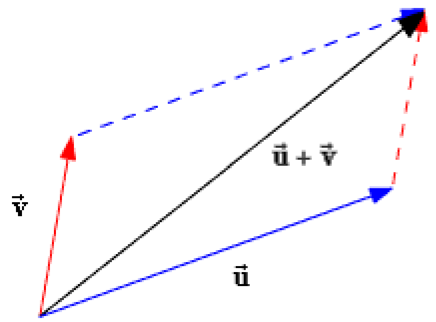
Vector Algebra Operations

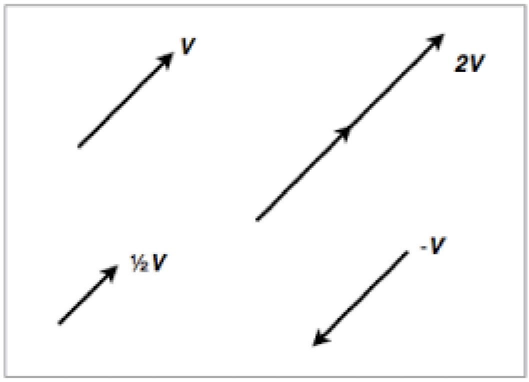
Definition (5):

Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ be two vectors and k is a scalar.

$(i)\ u + v = \langle u_1 + v_1, u_2 + v, u + v_3 \rangle$

$(ii)\ kv = \langle kv_1, kv_2, kv \rangle.$



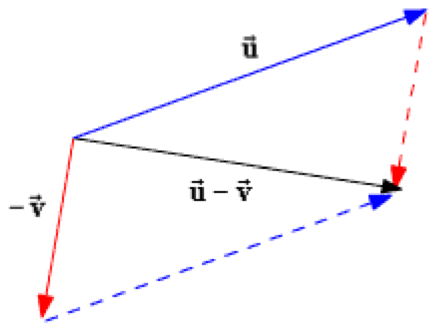


Note 2:

The length of the vector ku is the absolute value of the sclar k times the length of the vector u.That is $|k\,\vec{u}| = |k||\vec{u}|$

Note 3:

The difference $u - v$ of two vectors is defined by $u - v = u + (-v)$.



Properties of Vector Operations

Let u, v, w be vectors and α, β be scalars.

1: $u + v = v + u$ 2: $(u + v) + w = u + (v + w)$ 3: $u + O = u$

4: $u + (-u) = O$ 5: $Ou = O$ 6: $1.u = u$

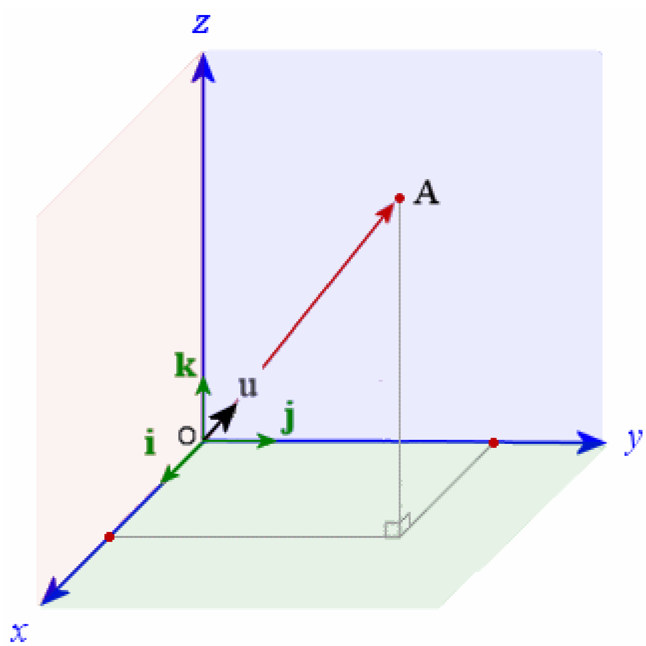
7: $\alpha(\beta u) = (\alpha\beta)u$ 8: $\alpha(u+v) = \alpha u + \alpha v$ 9: $(\alpha+\beta)u = \alpha u + \beta u$

Definition (6):

A vector v of length 1 is called *a unit vector*.

Definition (7):

The standard unit vectors are $i = \langle 1,0,0\rangle, j = \langle 0,1,0\rangle$ & $k = \langle 0,0,1\rangle$.



Note 4:

Any vector $v = \langle v_1, v_2, v_3\rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\begin{aligned} v = \langle v_1, v_2, v_3\rangle &= \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle \\ &= v_1\langle 1,0,0\rangle + v_2\langle 0,1,0\rangle + v_3\langle 0,0,1\rangle \\ &= v_1 i + v_2 j + v_3 k \end{aligned}$$

Note 5:

We call the scalars v_1 *the* $i-$*component*, v_2 *the* $j-$*component* and v_3 the $k-$*component* of the vector v.

Note 6:

The component form for the vector from the point $P_1(x_1, y_1, z_1)$ to the point $P_2(x_2, y_2, z_2)$ is given by

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k.$$

Definition (8):

The direction of nonzero vector v is $\frac{v}{|v|}$ which is a unit vector in the direction of v.

Example (3):

Find a unit vector u in the direction of the vector from the point $P_1(1,0,1)$ to the point $P_2(3,2,0)$.

Solution:

$$u = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{(3-1)i+(2-0)j+(0-1)k}{\sqrt{4+4+1}} = \frac{2i+2j-k}{\sqrt{9}} \quad = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k.$$

Note 7:

We can express any nonzero vector v in terms as its length times its direction. That is $v = |v|\frac{v}{|v|}$.

Example (4):

If $v = 3i - 4j$ is a velocity vector. Express v as a product of times its direction of a motion.

Solution:

Speed is the magnitude (length) of v;

$$|v| = \sqrt{9+16} = \sqrt{25} = 5.$$

The unit vector $\frac{v}{|v|}$ is the direction of v:

$$\frac{v}{|v|} = \frac{3}{5}i - \frac{4}{5}j$$

So

$$v = 3i - 4j = 5\left(\frac{3}{5}i - \frac{4}{5}j\right).$$

Length **Direction of motion**

Example (5):

A force of 6 newton is applied in the direction of the vector $v = 2i + 2j - k$. Express the force F as a product of its magnitude and direction.

Solution:

The force vector has magnitude 6 and direction $\frac{v}{|v|}$.

$\therefore F = 6\left(\frac{v}{|v|}\right) = 6\left(\frac{2i+2j-k}{\sqrt{4+4+1}}\right) = 6\left(\frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k\right).$

H.W. 1

1: Let $u = < 3, -2 >$ and $v = < -2, 5 >$. Find component form and length of each following vector.

$(i)\ 3u \quad (ii)\ u - v \quad (iii)\ -2u + 5v \quad (iv)\ \frac{3}{5}u + \frac{4}{5}v$.

2: Find the component form for the following:

(i) *The vector* $\overrightarrow{PQ}$ *, where* $P = (1, 3)$ *and* $Q = (2, -1)$.

(ii) *The vector from the point* $A = (2, 3)$ *to the origin*.

(iii) *The sum of* $\overrightarrow{AB}$ *and* $\overrightarrow{CD}$ *, where* $A = (1, -1), B = (2, 0), C = (-1, 3)$ *and* $D = (-2, 2)$.

3: Express each vector in the form $v = v_1 i + v_2 j + v_3 k$.

(i) $\overrightarrow{P_1P_2}$ *if* P_1 *is the point* $(5, 7, -1)$ *and* P_2 *is the point* $(2, 9, -2)$.

(ii) $\overrightarrow{AB}$ *if* A *is the point* $(-7, -8, 1)$ *and* B *is the point* $(-10, 8, 1)$.

(iii) $5u - v$ *if* $u = < 1, 1, -1 >$ *and* $v = < 2, 0, 3 >$.

4: Express each vector as a product of its length and direction.

$(a)\ 2i + j - 2k \quad (b)\ 9i - 2j + 6k \quad (c)\ \frac{3}{5}i + \frac{4}{5}k$.

5: Find the direction of $\overrightarrow{P_1P_2}$.

(i) $P_1(-1, 1, 5)$ *and* $P_2(2, 5, 0)$.

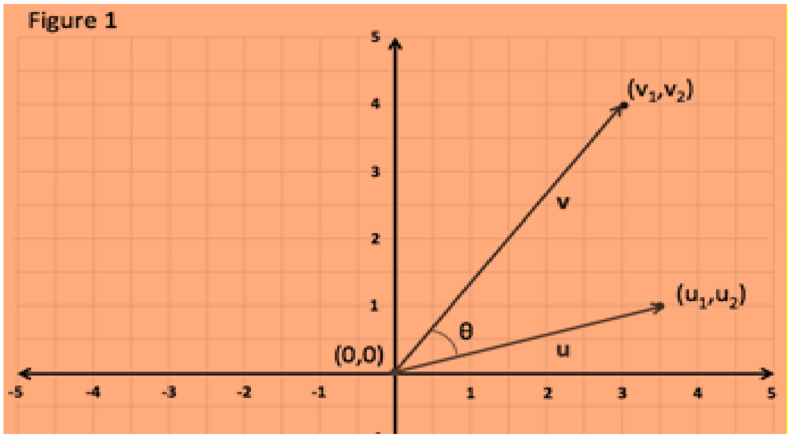
(ii) $P_1(1, 4, 5)$ *and* $P_2(4, -2, 7)$.

(iii) $P_1(0, 0, 0)$ *and* $P_2(2, -2, -2)$.

3: The Dot Product (Scalar Product)

Angle between Two Vectors:

When two nonzero vectors u and v are placed so their initial point coincide, they form an angle θ of measure $0 \leq \theta \leq \pi$.



Theorem (1): (Angle between Two Vectors)

The angle θ between two nonzero vectors $u = \langle u_1, u_2, u_3 \rangle$ & $v = \langle v_1, v_2, v_3 \rangle$ $is\ given\ by$ $\theta = \cos^{-1}\left(\frac{u_1v_1+u_2v_2+u_3v_3}{|u||v|}\right)$.

Definition (9):

$The\ dot\ product\ u \cdot v$ (“$u\ dot\ v$”) $of\ vectors\ u = \langle u_1, u_2, u_3 \rangle$ &
$v = \langle v_1, v_2, v_3 \rangle$ $is\ the\ scalar$ $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$.

Note 8:

Dot products are also called inner or scalar products because the product results is a scalar not a vector.

Example (6):

Find the dot product of the vectors u & v

$1: u = \langle 1, -2, -1 \rangle$ & $v = \langle -6, 2, -3 \rangle$.

$2: u = \frac{1}{2}i + 3j + k$ & $v = 4i - \frac{1}{2}j + \frac{2}{3}k$.

Solution:

$1: u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 = (1)(-6) + (-2)(2) + (-1)(-3) = -7.$

$2: u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 = \left(\frac{1}{2}\right)(4) + 3\left(-\frac{1}{2}\right) + (1)\left(\frac{2}{3}\right) = \frac{7}{6}.$

Definition (10):

The dot product of two nonzero vectors $\boldsymbol{u}$ & $\boldsymbol{v}$ *is given by*

$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\boldsymbol{\theta}$,

where $\boldsymbol{\theta}$ *is the angel between the two vectors* $\boldsymbol{u}$ & $\boldsymbol{v}$ *such that*

$0 \le \boldsymbol{\theta} \le \boldsymbol{\pi}$.

Note 9:

The angle $\boldsymbol{\theta}$ between the two nonzero vectors $\boldsymbol{u}$ & $\boldsymbol{v}$ is given by the formula

$$\boldsymbol{\theta} = \cos^{-1}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}\right).$$

Example (7):

Find the angle between the two vectors $\boldsymbol{u} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ & $\boldsymbol{v} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.

Solution:

$$\boldsymbol{\theta} = \cos^{-1}\left(\frac{u\cdot v}{|u||v|}\right) = \cos^{-1}\left(\frac{u_1v_1+u_2v_2+u_3v_3}{\sqrt{u_1^2+u_2^2+u_3^2}\ \sqrt{v_1^2+v_2^2+v_3^2}}\right) = \cos^{-1}\left(\frac{6-6+4}{\sqrt{9}\ \sqrt{49}}\right) = \cos^{-1}\frac{4}{21}$$

$$\approx 79^\circ .$$

Note 10:

The angle $\boldsymbol{\theta}$ between the two nonzero vectors $\boldsymbol{u}$ & $\boldsymbol{v}$ is *acute angle* if $\boldsymbol{u}\cdot\boldsymbol{v} > 0$ *and obtuse angle if* $\boldsymbol{u}\cdot\boldsymbol{v} < 0$.

Example (8):

Find the angle $\boldsymbol{\theta}$ in the triangle ABC determined by the vertices

$\boldsymbol{A}(3,5)$, $\boldsymbol{B}(5,2)$ & $\boldsymbol{C}(0,0)$.

Solution:

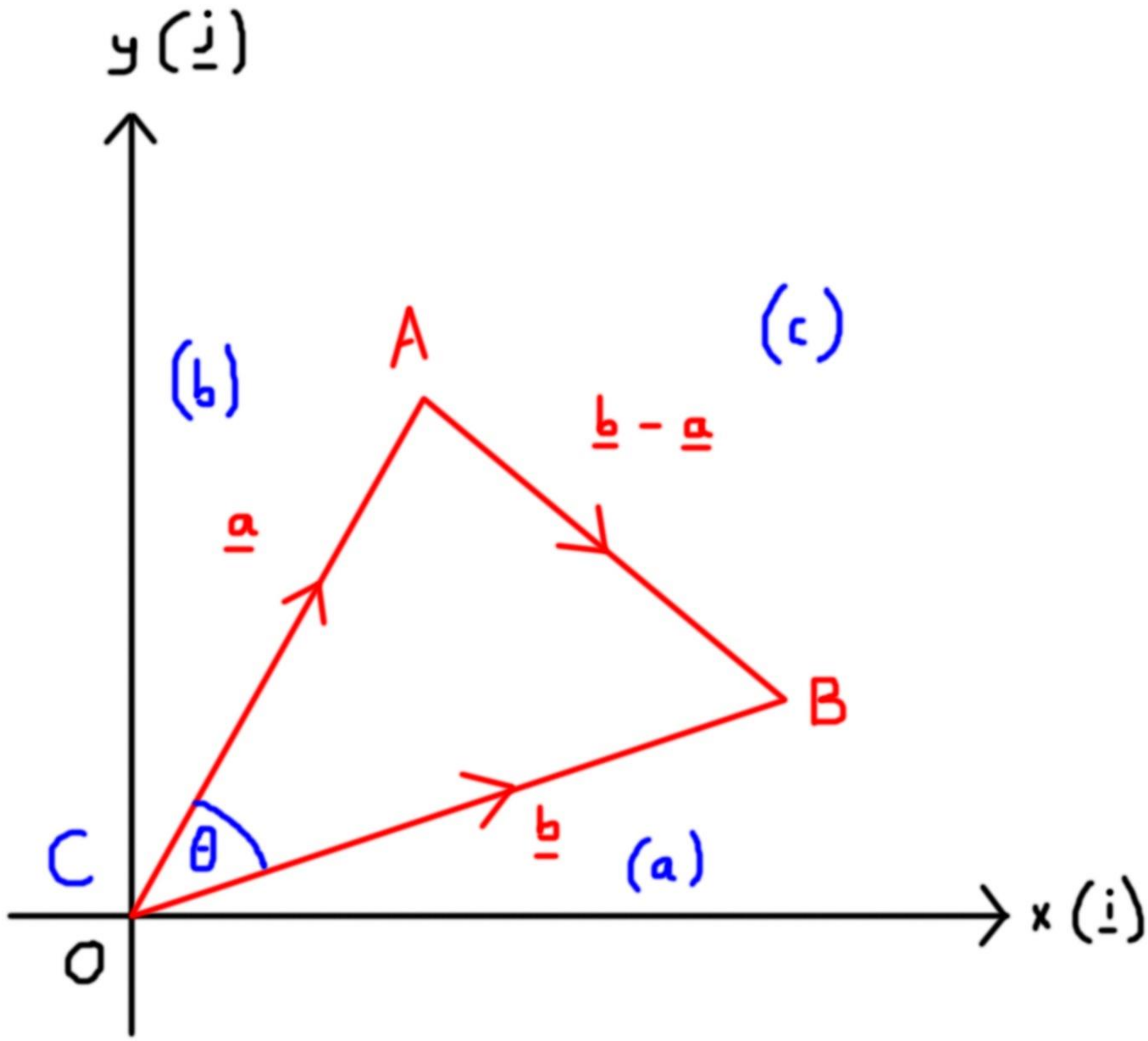
The angle $\boldsymbol{\theta}$ is the angle between the two vectors $\overrightarrow{CA}$ & $\overrightarrow{CB}$.

$\therefore \overrightarrow{CA} = 3\boldsymbol{i} + 5\boldsymbol{j}$ & $\overrightarrow{CB} = 5\boldsymbol{i} + 2\boldsymbol{j}$.

$(\overrightarrow{CA})\cdot(\overrightarrow{CB}) = (3)(5) + (5)(2) = 25.$

$|\overrightarrow{CA}| = \sqrt{9+25} = \sqrt{34}$, $|\overrightarrow{CB}| = \sqrt{25+4} = \sqrt{29}$.

$$\therefore \boldsymbol{\theta} = \cos^{-1}\left(\frac{(\overrightarrow{CA})\cdot(\overrightarrow{CB})}{|\overrightarrow{CA}|\ |\overrightarrow{CB}|}\right) = \cos^{-1}\left(\frac{25}{\sqrt{34}\sqrt{29}}\right) = \cos^{-1}\left(\frac{25}{\sqrt{986}}\right) \approx 37^\circ .$$



Orthogonal (Perpendicular) Vectors

Two nonzero vectors u & v are orthogonal (perpendicular) if the angle between them $\frac{\pi}{2}$.

$\therefore u \cdot v = |u||v| \cos\theta = |u||v| \cos\frac{\pi}{2} = 0.$

The converse is also true. If u & v are nonzero vectors with $u \cdot v = 0 \rightarrow \theta = \frac{\pi}{2}$.

$\therefore$Two nonzero vectors u & v are orthogonal if and only if $u \cdot v = 0$.

Definition (11):

Two nonzero vectors u & v are orthogonal if $u \cdot v = 0$.

Example (9):

Determine if the two given vectors are orthogonal.

$1: \boldsymbol{u} = \langle 3, -2 \rangle \;\&\; \boldsymbol{v} = \langle 4, 6 \rangle$.

$2: \boldsymbol{u} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} \;\&\; \boldsymbol{v} = 2\boldsymbol{j} + 4\boldsymbol{k}.$

$3: \boldsymbol{u} = 2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} \;\&\; \boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}.$

Solution:

$1: \boldsymbol{u} \cdot \boldsymbol{v} = 12 - 12 = 0 \;\rightarrow \boldsymbol{u} \,\&\, \boldsymbol{v}$ ***are orthogonal.***

$2: \boldsymbol{u} \cdot \boldsymbol{v} = 0 - 4 + 4 = 0 \;\rightarrow\; \boldsymbol{u} \,\&\, \boldsymbol{v}$ ***are orthogonal.***

$3: \boldsymbol{u} \cdot \boldsymbol{v} = 2 - 2 - 6 = -6 \neq 0 \;\rightarrow\; \boldsymbol{u} \,\&\, \boldsymbol{v}$ ***are not orthogonal.***

Properties of Dot Product

If $\boldsymbol{u}, \boldsymbol{v} \,\&\, \boldsymbol{w}$ **are any vectors and** $\boldsymbol{c}$ **is scalar, then:**

$1: \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}.$

$2: (\boldsymbol{c}\,\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{c}\boldsymbol{v}) = \boldsymbol{c}(\boldsymbol{u} \cdot \boldsymbol{v}).$

$3: \boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$

$4: \boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}|^2$.

$5: \boldsymbol{O} \cdot \boldsymbol{u} = 0,$ ***where O is zero vector and* 0 *is zero number*.**

Definition (12):

The work done by a constant force $\boldsymbol{F}$ **acting through a displacement** $\boldsymbol{D}$ **is**

$\boldsymbol{work} = \boldsymbol{F} \cdot \boldsymbol{D}$.

Example (10):

If $|\boldsymbol{F}| = 40$ ***Newtons***, $|\boldsymbol{D}| = 3$ ***meters*** $\&\; \boldsymbol{\theta} = 60°$. **Find the work done.**

Solution:

$$\boldsymbol{work} = \boldsymbol{F} \cdot \boldsymbol{D} = |\boldsymbol{F}||\boldsymbol{D}| \cos \boldsymbol{\theta} = (40)(3) \cos 60° = 60\; \boldsymbol{Joules}.$$

H.W. 2

1: Find $\boldsymbol{v}.\boldsymbol{u}$, $|\boldsymbol{u}|, |\boldsymbol{v}|$ **and the cosine angle between** $\boldsymbol{u}$ ***and*** $\boldsymbol{v}$**.**

$(\boldsymbol{a})\; \boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j} + \sqrt{5}\boldsymbol{k}\;, \boldsymbol{u} = -2\boldsymbol{i} + 4\boldsymbol{j} - \sqrt{5}\boldsymbol{k}$.

$(\boldsymbol{b})\; \boldsymbol{v} = 2\boldsymbol{i} + 10\boldsymbol{j} - 4\boldsymbol{k},\;\; \boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

$(\boldsymbol{c})\; \boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j}\,,\; \boldsymbol{u} = \sqrt{2}\boldsymbol{i} - \sqrt{3}\boldsymbol{j} + 2\boldsymbol{k}$.

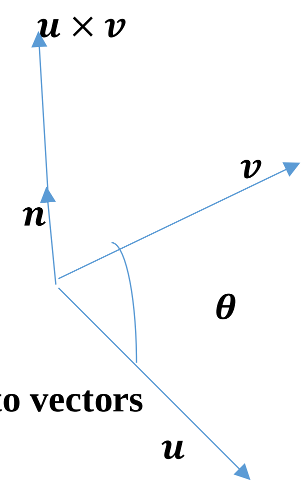
2: Find the angle between two vectors.

$(\boldsymbol{a})\; \boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j}\,, \boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \quad (\boldsymbol{b})\; \boldsymbol{u} = \boldsymbol{i} + \sqrt{2}\boldsymbol{j} - \sqrt{2}\boldsymbol{k}\,, \boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

3: Find the measures of the angles of the triangle whose vertices $\boldsymbol{A}(-1, 0)$,
$\boldsymbol{B}(2, 1)$ ***and*** $\boldsymbol{C}(1, -2)$.

4: Cross Product (Vector Product):

Let $\boldsymbol{u}$ & $\boldsymbol{v}$ be two nonzero vectors in space. If $\boldsymbol{u}$ & $\boldsymbol{v}$ are not parallel, they determined a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane. This means that we choose $\boldsymbol{n}$ to be unit (normal) vector.



Definition (13):

The cross product $\boldsymbol{u} \times \boldsymbol{v}$ ("$\boldsymbol{u}$ *cross* $\boldsymbol{v}$") *is the vector*

$\boldsymbol{u} \times \boldsymbol{v} = ((|\boldsymbol{u}||\boldsymbol{v}|)\sin\boldsymbol{\theta})\boldsymbol{n}$.

Note (11):

Cross products are also called vector products Because it is a vector, and can be applied only to vectors *in space*.

Note (12):

Nonzero vectors $\boldsymbol{u}$ & $\boldsymbol{v}$ are parallel if and only if $\boldsymbol{u} \times \boldsymbol{v} = 0$.

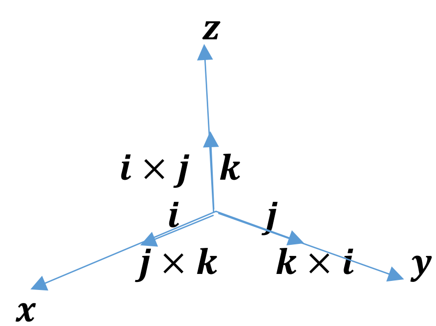
Properties of the Cross Product:

If $\boldsymbol{u}, \boldsymbol{v}$ & $\boldsymbol{w}$ are any vectors and $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ *are scalars, then*:

1) $(\boldsymbol{\alpha u}) \times (\boldsymbol{\beta v}) = (\boldsymbol{\alpha}\,\boldsymbol{\beta})(\boldsymbol{u} \times \boldsymbol{v})$
2) $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
3) $\boldsymbol{v} \times \boldsymbol{u} = -(\boldsymbol{u} \times \boldsymbol{v})$
4) $(\boldsymbol{v} + \boldsymbol{w}) \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u}$
5) $\boldsymbol{O} \times \boldsymbol{u} = \boldsymbol{O}$
6) $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$

Note (13):

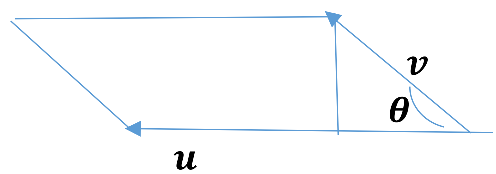
1: $\boldsymbol{i} \times \boldsymbol{j} = -(\boldsymbol{j} \times \boldsymbol{i}) = \boldsymbol{k}$
2: $\boldsymbol{j} \times \boldsymbol{k} = -(\boldsymbol{k} \times \boldsymbol{j}) = \boldsymbol{i}$
3: $\boldsymbol{k} \times \boldsymbol{i} = -(\boldsymbol{i} \times \boldsymbol{k}) = \boldsymbol{j}$
4: $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{O}$.



The Area of a Parallelogram:

The area of the parallelogram determined by $the\ two\ vectors\ u\ \&\ v$**, with base** $|u|$ **and height** $|v||\sin\theta|$

$A = |u||v|\sin\theta$.



Note (14):

If $u = u_1 i + u_2 j + u_3 k$ & $v = v_1 i + v_2 j + v_3 k$ **, then**

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example (11):

If $u = 2i + j + k$ & $v = -4i + 3j + k$ **. Find** $u \times v$ & $v \times u$**.**

Solution:

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = -2i - 6j + 10k .$$

$v \times u = 2i + 6j - 10k$.

Example (12):

Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ & $R(-1,1,2)$.

Solution:

The perpendicular vector to the plane PQR **is**

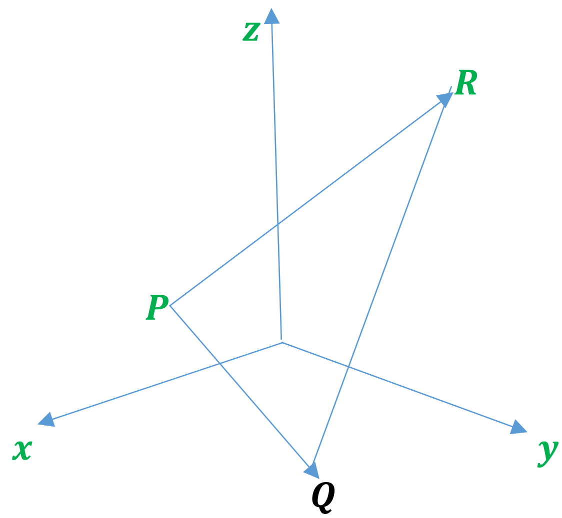
$\overrightarrow{PQ} \times \overrightarrow{PR}$.

$\overrightarrow{PQ} = i + 2j - k$

$\overrightarrow{PR} = -2i + 2j + 2k$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= 6i + 6k .$$



Note (15):

The area of the triangle with u & v are two sides is given by
$A = \frac{1}{2}|u \times v|$.

Example (13):

Find the area of the triangle with vertices $P(1,-1,0), Q(2,1,-1)$ & $R(-1,1,2)$.

Solution:

The area of the triangle is:

$$A = \frac{1}{2}\,|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{36+36} = 3\sqrt{2}\ \textit{square units}.$$

Example (14):

Find a unit vector perpendicular to the plane of $P(1,-1,0), Q(2,1,-1)$ & $R(-1,1,2)$.

Solution:

The unit perpendicular vector to the plane PQR **is**

$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6i+6k}{6\sqrt{2}} = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j.$$

H.W.3

1: Find the length and direction (when defined) of $u \times v$ *and* $v \times u$.

- (a) $u = 2i - 2j - k, v = i - k.$
- (b) $u = 2i - 2j + 4k, v = -i + j - 2k.$
- (c) $u = 2i, v = -3j$
- (d) $u = -8i - 2j - 4k, v = 2i + 2j + k.$

2: Find the areas of the triangle determined by the points P, Q *and* R.
Then find a unit vector perpendicular to plane PQR.

- (a) $P(1,-1,2), Q(2,0,-1), R(0,2,1).$
- (b) $P(2,-2,1), Q(3,-1,2), R(3,-1,1).$
- (c) $P(-2,2,0), Q(0,1,-1), R(-1,2,-2).$

3: Find the area of parallelogram whose vertices:

- (a) $A(1,0), B(0,1), C(-1,0), D(0,-1).$
- (b) $A(-1,2), B(2,0), C(7,1), D(4,3).$

Lecture 3

Trigonometric Functions

Trigonometry, as the word implies, is concerned with the measurements of the parts of a triangle.

1: The Angle

Definition (1): (A ray or Half-Line)

A ray **or half-line is that portion of a line that starts at a point V on the line and extends indefinitely in one direction. The starting point V of a ray is called its vertex.**

Definition (2): (Angle)

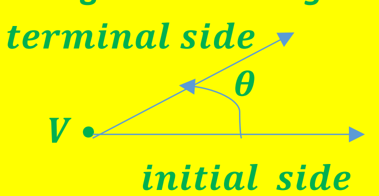
If two rays are drawn with a common vertex, they form an *angle*. We call one ray of an angle the *initial side* and the other the *terminal side.*

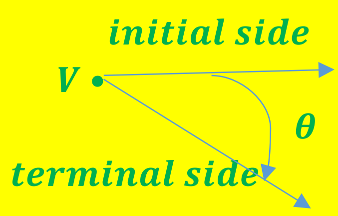
Note 1:

1: The angle is generated by revolving a ray from the initial side to a terminal side.

2: The angle is called *positive* if the direction of rotation is counterclockwise.

3: The angle is called *negative* if the direction of rotation is clockwise.





θ *is positive*

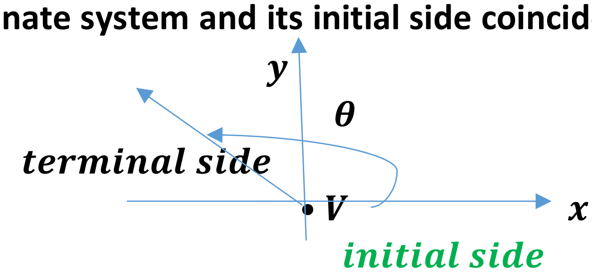
Counterclockwise

θ *is negative*

Clockwise

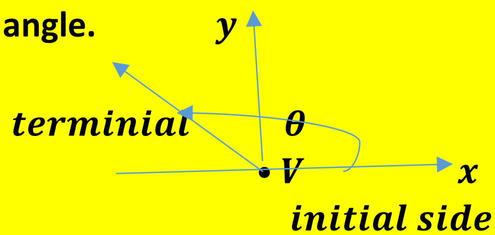
Definition (3): (Angle in Standard Position)

An angle θ is said to be in *standard position* if its vertex at the origin of a rectangular coordinate system and its initial side coincides with the positive $x-axis$.

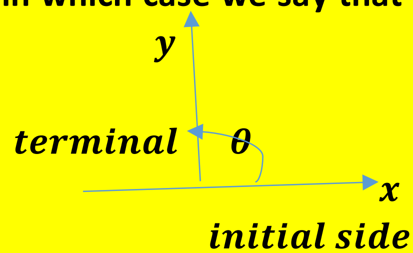


Note 2:

When an angle θ is in standard position, the terminal side will lie either in a quadrant, in which case we say that θ lies in that quadrant, or the terminal side will lie on the $x-axis\ or\ y-axis$, in which case we say that θ is quadrantile angle.



θ *lies in the 2nd quadrant*



θ *is quadrantile angle*

2: Measure of Angles

We measure angles by determining the amount of rotation ended for the initial side to become coincident with terminal side. The two commonly used measures for angles are *degrees and radians*.

First: Degrees

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it coincides with itself (one revolution) is said to measure 360 degrees ($360°$), then $1° = \frac{1}{360}\ revolution$.

Note 3:

1: The subdivisions of degree are minutes and seconds, where

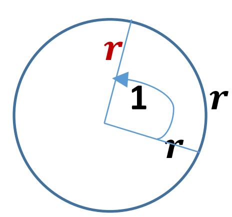
$$1\ minute = 1' = \frac{1}{60}\ degree\ ,1\ second = 1'' = \frac{1}{60}\ minute = \frac{1}{3600}\ degree.$$

$$2: 1\ counterclockwise\ revolution = 360°\ .$$

$$1° = 60'\quad ,\quad 1' = 60''\ .$$

Second: Radians

A radian (rad.) is defined as the measure of $the\ central\ angle$ subtended by an arc $of\ a\ circle\ equal\ to\ the\ radius\ of\ the\ circle.$



Example (1):

Convert each angle in degrees to radians.

$60°\ ,150°\ ,-45°\ ,90°\ ,105°\ .$

Solution:

$$60° = 60°\left(\frac{\pi}{180°}\right) = \frac{\pi}{3}\ rad.$$

$$150° = 150°\left(\frac{\pi}{180°}\right) = \frac{5\pi}{6}\ rad.$$

$$-45° = -45°\left(\frac{\pi}{180°}\right) = -\frac{\pi}{4}\ rad.$$

$$90° = 90°\left(\frac{\pi}{180°}\right) = \frac{\pi}{2}\ rad.$$

$$105° = 105°\left(\frac{\pi}{180°}\right) = \frac{7\pi}{12}\ rad.$$

Example (2):

Convert each angle in radians to degrees.

$\frac{\pi}{6}\ rad.\ ,\frac{3\pi}{2}\ rad.,-\frac{3\pi}{4}\ rad.,\frac{7\pi}{3}\ rad.,-\frac{\pi}{3}\ rad..$

Solution:

$$\frac{\pi}{6}\ rad. = \frac{\pi}{6}\ rad.\left(\frac{180°}{\pi}\right) = 30°.$$

$$\frac{3\pi}{2}\ rad. = \frac{3\pi}{2}\ rad.\left(\frac{180°}{\pi}\right) = 270°.$$

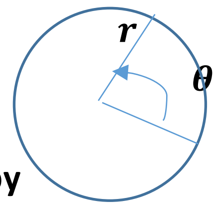
$$-\frac{3\pi}{4}\ rad. = -\frac{3\pi}{4}\ rad.\left(\frac{180°}{\pi}\right) = -135°.$$

$$\frac{7\pi}{3}\ rad. = \frac{7\pi}{3}\ rad.\left(\frac{180°}{\pi}\right) = 420°.$$

$$-\frac{\pi}{3}\ rad. = -\frac{\pi}{3}\ rad.\left(\frac{180°}{\pi}\right) = -60°.$$

Theorem (1):

For a $circle\ of\ radius\ r$, a central angle of θ radians $subtended\ an\ arc$ whose length $s\ is$: $s = r\theta$.



Example (3):

Find the length of the arc of a circle of radius $2cm$ subtended by $a\ central$ angle of $0.25\ rad.$

Solution:

$$s = r\theta = 2(0.25) = 0.5\ cm.$$

Example (4):

If the radius of a $circle\ is\ 5\ cm$. What angle is subtended by an $arc\ of\ 20\ cm$?

Solution:

$$\theta = \frac{s}{r} = \frac{20}{5} = 4\ rad.$$

Example (5):

What is the radius of the circle where the central angle $5\ rad.$ subtended by an arc of $2\ cm$?

Solution:

$$r = \frac{s}{\theta} = \frac{2}{5} = 0.4\ cm\ .$$

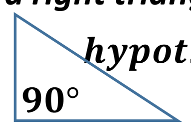
3: Right Triangle Trigonometry

Definition (4): (Right Triangle)

A triangle in which one angle is a right $(90^\circ\ or\ \frac{\pi}{2}\ rad.)$ is called ***a right triangle***.

The side opposite the right angle is called the *hypotenuse*.

The remaining two sides are called the legs of the triangle.



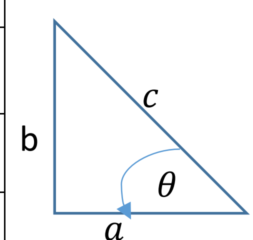
Definition (5): (Acute Angle)

The angle θ is called an acute angle if $0^\circ < \theta < 90^\circ$ or $\left(0 < \theta < \frac{\pi}{2}\right) rad.$

Definition (6):

The six ratios of the length of the sides of a right triangle are called trigonometric functions of acute angles and are defined as follows:

No.	Function Name	Abbreviation	Value
1	*Sine of* θ	$\sin\theta$	$\frac{b}{c}$
2	*cosin of* θ	$\cos\theta$	$\frac{a}{c}$
3	*tangent of* θ	$\tan\theta$	$\frac{b}{a}$
4	*cosecant of* θ	$\csc\theta$	$\frac{c}{b}$
5	*secant of* θ	$\sec\theta$	$\frac{c}{a}$
6	*cotangent of* θ	$\cot\theta$	$\frac{a}{b}$



H.W.1

Q1: Convert the angle from degree measure into radian measure.

$240°, 135°, -270°, -315°, 150°, -225°$.

Q2: Convert the angle from radian measure into degree measure.

$\pi, -\frac{2\pi}{3}, \frac{7\pi}{6}, -\frac{5\pi}{3}, \frac{11\pi}{6}, -\frac{\pi}{6}$.

Q3: A circle has radius $4\ cm$. Find the length of the arc intercepted by a central angle of $240°$.

Q4: A circle has radius $12\ cm$. Find the length of the arc intercepted by a central $angle\ of\ \frac{3\pi}{4}\ rad$.

Q5: Find the radius of the circle with an arc $8\ cm$ and central angle $330°$.

Q6: Find the radius of the circle with an arc $3\ cm$ and central angle $\frac{4\pi}{3}\ rad$..

4: Fundamental Identities

1: Reciprocal Identities:

$(a)\ \csc\theta = \frac{1}{\sin\theta}$, $(b)\ \sec\theta = \frac{1}{\cos\theta}$, $(c)\ \cot\theta = \frac{1}{\tan\theta}$.

2: Quotient Identities:

$(a)\ \tan\theta = \frac{\sin\theta}{\cos\theta}$, $(b)\ \cot\theta = \frac{\cos\theta}{\sin\theta}$.

3: Pythagorean Identities:

(a) $cos^2\theta + sin^2\theta = 1$, $(b)\ 1 + tan^2\theta = sec^2\theta$, $(c)\ cot^2\theta + 1 = csc^2\theta$.

4: Double Angle Formals

$(a)\ \sin 2\theta = 2\sin\theta\cos\theta$.

$(b)\ \cos 2\theta = cos^2\theta - sin^2\theta = 1 - 2sin^2\theta = 2cos^2\theta - 1$.

$(c)\ \tan 2\theta = \frac{2\tan\theta}{1-tan^2\theta}$.

5: Half Angle Formulas

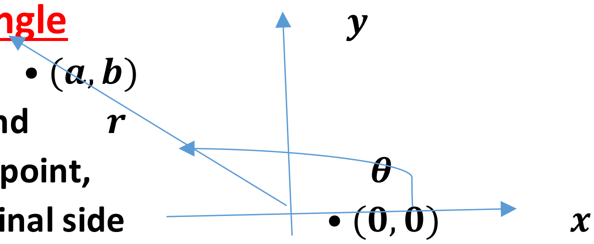
$(a)\ sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \rightarrow \sin\frac{\theta}{2} = \mp\sqrt{\frac{1-\cos\theta}{2}}$.

$(b)\ cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \rightarrow \cos\frac{\theta}{2} = \mp\sqrt{\frac{1+\cos\theta}{2}}$.

$(c)\ tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} \rightarrow \tan\frac{\theta}{2} = \mp\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$.

5: Trigonometric Functions for Any Angle

Definition (7):



Let θ be any angle in standard position, and *let* (a, b) *denote* the coordinates of any point, *expected the origin* $(0,0)$, on the terminal side *of* θ.

If $r = \sqrt{a^2+b^2}$ denotes the distance from $(0,0)$ *to* (a,b). Then the six trigonometric functions of θ are defined as the ratios:

$$\sin\theta = \frac{b}{r}, \cos\theta = \frac{a}{r}, \tan\theta = \frac{b}{a}, \csc\theta = \frac{r}{b}, \sec\theta = \frac{r}{a}, \cot\theta = \frac{a}{b}.$$

Provided no denominator equals zero. If a denominator equals zero, the trigonometric function of the angle θ is not defined.

Example (28):

Find the exact value of each of the six trigonometric functions of a positive angle θ if $(4,-3)$ is a point on its terminal side.

Solution:

$$r = \sqrt{a^2+b^2} = \sqrt{16+9} = \sqrt{25} = 5 .$$

$$\therefore \sin\theta = \frac{b}{r} = -\frac{3}{5}, \cos\theta = \frac{a}{r} = \frac{4}{5}, \tan\theta = \frac{b}{a} = -\frac{3}{4},$$

$$\csc\theta = \frac{r}{b} = -\frac{5}{3}, \sec\theta = \frac{r}{a} = \frac{5}{4}, \cot\theta = \frac{a}{b} = -\frac{4}{3}.$$

Note 4:

$\theta(Rad.)$	$\theta(Degree)$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$0°$	0	1	0	*Not Defined*	1	*Not Defined*
$\frac{\pi}{2}$	$90°$	1	0	*Not Defined*	1	*Not Defined*	0
π	$180°$	0	-1	0	*Not Defined*	-1	*Not Defined*
$\frac{3\pi}{2}$	$270°$	-1	0	*Not Defined*	-1	*Not Defined*	0

Note 5:

The signs of the trigonometric functions of an angle in a given quadrant are given below:

$Quadrant\ of\ \theta$	$\sin\theta, \csc\theta$	$\cos\theta, \sec\theta$	$\tan\theta, \cot\theta$
First	*Positive*	*Positive*	*Positive*
Second	*Positive*	*Negative*	*Negative*
Third	*Negative*	*Negative*	*Positive*
Forth	*Negative*	*Positive*	*Negative*

Definition (8): (Unit Circle)

The unit circle is a circle whose radius 1 unit and whose center is at the origin of a rectangular coordinate system.

Definition (9):

Let t be a real number and let $P(a,b)$ be a point on the unit circle that corresponds $to\ t$. Then

$$\sin t = b, \quad \cos t = a, \tan t = \frac{b}{a}, a \neq 0, \csc t = \frac{1}{b}, b \neq 0, \sec t = \frac{1}{a}, a \neq 0,$$
$$\cot t = \frac{a}{b}, b \neq 0.$$

Example (34):

Find the values of $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, $\cot t$ if $P(-\frac{1}{2},\frac{\sqrt{3}}{2})$ is the point on the unit circle that corresponds to the real number t.

Solution:

Here, $a = -\frac{1}{2}, b = \frac{\sqrt{3}}{2}$.

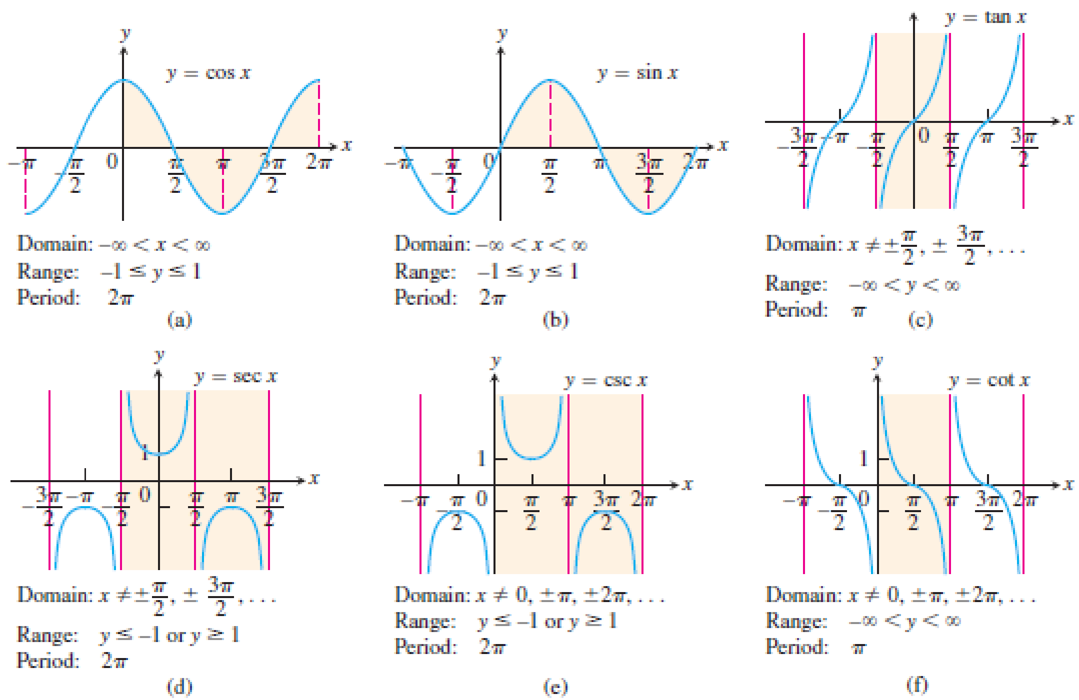
$$\therefore \sin t = b = \frac{\sqrt{3}}{2}, \ \cos t = a = -\frac{1}{2}, \ \tan t = \frac{b}{a} = -\sqrt{3}.$$
$$\csc t = \frac{1}{b} = \frac{2}{\sqrt{3}}, \sec t = \frac{1}{a} = -2, \cot t = \frac{a}{b} = -\frac{1}{\sqrt{3}}.$$

6: The Domain and the Range of the Trigonometric Functions

The Function	*Domain*	*Range*
$\sin\theta$	$All\ Real\ Numbers\ R$	$[-1,1]$
$\cos\theta$	$All\ Real\ Numbers\ R$	$[-1,1]$
$\tan\theta$	$R/\{\theta = \mp\left(\frac{2n+1}{2}\right)\pi, n = 0, \mp1, \mp2, \mp3, \cdots\}$	$All\ Real\ Numbers\ R$
$\csc\theta$	$R/\{\theta = \mp n\pi, n = 0, \mp1, \mp2, \mp3, \cdots\}$	$R/(-1,1)$

$\sec\theta$	$R/\{\theta = \mp\left(\frac{2n+1}{2}\right)\pi,\ n = 0, \mp1, \mp2, \mp3, \cdots\}$	$R/(-1,1)$
$\cot\theta$	$R/\{\theta = \mp n\pi,\ n = 0, \mp1, \mp2, \mp3, \cdots\}$	$All\ Real\ Numbers\ R$

7: Graph of Trigonometric Functions:



Definition (10): (Periodic Function)

A function f is called periodic if there is a positive number p such that whenever θ is in $the\ domain\ of\ f$, so is $\theta + p$, and $f(\theta + p) = f(\theta)$.

If there is a smallest $such\ number\ p$, this smallest value is called the period of f.

Note 6:

The trigonometric functions are periodic as follows:

1: $\sin(\theta + 2\pi) = \sin\theta$ 4: $\csc(\theta + 2\pi) = \csc\theta$

2: $\cos(\theta + 2\pi) = \cos\theta$ 5: $\sec(\theta + 2\pi) = \sec\theta$

3: $\tan(\theta + \pi) = \tan\theta$ 6: $\cot(\theta + \pi) = \cot\theta$

Definition (11): (Even Function)

A function f is even if $f(-\theta) = f(\theta)$ for all θ in the domain of f .

Definition (12): (Odd Function)

A function f is odd if $f(-\theta) = -f(\theta)$ for all θ in the domain of f .

Theorem (2):

(i) *The trigonometric functions* $\cos\theta$ & $\sec\theta$ *are even functions.*

(ii) *The trigonometric functions* $\sin\theta, \tan\theta, \csc\theta, \& \cot\theta$ *are odd functions.*

H.W.2

Q1: A point on the terminal side of an angle θ in standard position is given. Find *exact value* of each of the six trigonometric functions of θ .

$(2,-2), (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Q2: Name the quadrant in which the angle θ lies.

$1:\ \sin\theta > 0, \cos\theta < 0 \quad 2:\ \sin\theta < 0, \cos\theta > 0 \quad 3:\ \sin\theta < 0, \tan\theta < 0$

$4:\ \cos\theta > 0\ ,\ \tan\theta > 0 \quad 5:\ \sec\theta < 0, \tan\theta > 0 \quad 6:\ \cos\theta > 0\ , \cot\theta < 0$

Q3: Find the exact values of each of remaining five trigonometric functions of the acute angle θ.

$\sin\theta = \frac{\sqrt{3}}{4}, \cos\theta = \frac{1}{3}, \tan\theta = \frac{1}{2}, \csc\theta = 5, \sec\theta = \frac{5}{3}, \cot\theta = 2$.

Q4: The point P on the unit circle that corresponds to a real number t is given. Find the trigonometric functions of t.

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$.

Lecture 4

Limits of Functions

1: Introduction:

Suppose that f is a function $such\ that\ f:(a,c)\cup(c,b)\ \rightarrow\ R$. We wish to consider the behavior of f as the variable $x\ approaches\ c$. If $f(x)$ approaches a particular finite value L as $x\ approaches\ c$, then we say that the function f has the limit L as $x\ approaches\ c.\ We\ write\ \lim_{x\to c} f(x)=L$.

2: One-Sided Limit:

We say that $\lim_{x\to c^-} f(x)=L$ if the values of f become closer and closer to L when x is near to c but on the left. In other words, in $studying\ \lim_{x\to c^-} f(x)$, we only consider values of x that are less than c.

Likewise, We say that $\lim_{x\to c^+} f(x)=L$ if the values of f become closer and closer to L when x is near to c but on the right. In other words, in $studying$ $\lim_{x\to c^+} f(x)$, we only consider values of x that are greater than c.

3: Properties of Limit:

Theorem 1:

Let $f\ and\ g$ are two functions, c is a real number and $\lim_{x\to c} f(x)=L_1\ and$ $\lim_{x\to c} g(x)=L_2$. Then:

1: $\lim_{x\to c}[f(x)\mp g(x)]=\lim_{x\to c} f(x)\mp\lim_{x\to c} g(x)=L_1\mp L_2$.

2: $\lim_{x\to c}[f(x).g(x)]=\lim_{x\to c} f(x).\lim_{x\to c} g(x)=L_1.L_2$.

3: $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}=\frac{L_1}{L_2}$ $,provided\ that\ L_2\neq 0$.

4: $\lim_{x\to c}[\alpha f(x)]=\alpha\,[\lim_{x\to c} f(x)]=\alpha\, L_1$.

Theorem 2:

If $\lim_{x \to c} f(x)$ **exists, then** its **unique.**

Theorem 3:

If $\lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} f(x)$ **either does not exist or exists but not zero, then**

$\lim_{x \to c} \frac{f(x)}{g(x)}$ $does\ not\ exist.$

Theorem 4: (Pinching or Sandwich or Squeeze Theorem)

Suppose that f, g and h **are functions whose domains each contain**

$S = (a, c) \cup (c, b)$. **Assume further that** $g(x) \leq f(x) \leq h(x)$ **for all** $x \in S$.

If $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} h(x) = L$. **Then** $\lim_{x \to c} f(x) = L$.

Theorem 5:

$\lim_{x \to c} f(x) = L$ $if\ and\ only\ if$ $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$

Theorem 6:

$\lim_{x \to c}[f(x)]^n = [\lim_{x \to c} f(x)]^n, where\ n\ is\ positive\ integer.$

Theorem 7:

$\lim_{x \to c} k = k$ $, where\ k\ is\ a\ constant.$

Theorem 8:

$\lim_{x \to c} x = c$.

Theorem 9:

$\lim_{x \to c} x^n = c^n$ $, where\ n\ is\ positive\ integer.$

Theorem 10:

$\lim_{x \to c} \sqrt[n]{x} =$
$\sqrt[n]{c}$ $, where\ n\ is\ positive\ integer. (If\ n\ is\ even\ , we\ assume\ c > 0).$

Theorem 11:

$\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)}$, *where n is a positive integer*. (*If n is even*, *we assume that* $\lim\limits_{x \to c} f(x) > 0$).

Example 1:

Evaluate the following limits.

1: $\lim\limits_{x \to 5}(2x^2 - 3x + 4)$ **2**: $\lim\limits_{x \to -2} \frac{x^3+2x^2-1}{5-3x}$.

Solution:

$$1: \lim_{x \to 5}(2x^2 - 3x + 4) = 2\lim_{x \to 5} x^2 - 3\lim_{x \to 5} x + \lim_{x \to 5} 4 = 2(5)^2 - 3(5) + 4$$

$$= 50 - 15 + 4 = 39.$$

$$2: \lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim\limits_{x \to -2}(x^3 + 2x^2 - 1)}{\lim\limits_{x \to -2}(5 - 3x} = \frac{-8 + 8 - 1}{5 + 6} = \frac{-1}{11}.$$

Note 1:

If f is a polynomial or rational function and c is in the domain of f. Then

$\lim\limits_{x \to c} f(x) = f(c)$.

Example 2:

Find $\lim\limits_{x \to 1} \frac{x^2-1}{x-1}$.

Solution:

Let $f(x) = \frac{x^2-1}{x-1}$. **We cannot find the limit by substituting $x = 1$ because $f(1)$ is not defined. Since the denominator is zero at $x = 1$. In this case, we need to do some preliminary algebra. We factor the numerator as a difference of squares and cancel the common the common factor, then compute the limit as follow:**

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)}=\lim_{x\to 1}(x+1)=1+1=2.$$

Example 2:

$Find\ \lim_{x\to 2}\frac{x-2}{x^2+x-6}$.

Simplify

$$Find\ \lim_{x\to 2}\frac{x-2}{x^2+x-6}=\lim_{x\to 2}\frac{(x-2)}{(x-2)(x+3)}=\lim_{x\to 2}\frac{1}{x+3}=\frac{1}{2+3}=\frac{1}{5}.$$

Example 3:

$$\lim_{x\to -3}\frac{x+3}{x^3+27}=\lim_{x\to -3}\frac{x+3}{(x+3)(x^2-3x+9)}=\lim_{x\to -3}\frac{1}{x^2-3x+9}=\frac{1}{9+9+9}=\frac{1}{27}.$$

Example 4:

$Find\ \lim_{x\to 0}\frac{(3+x)^2-9}{x}$.

Solution:

We simplify $f(x)=\frac{(3+x)^2-9}{x}$ algebraically the find the limit as follws:

$$f(x)=\frac{(3+x)^2-9}{x}=\frac{9+6x+x^2-9}{x}=\frac{6x+x^2}{x}.$$

Then $\lim_{x\to 0}\frac{(3+x)^2-9}{x}=\lim_{x\to 0}\frac{9+6x+x^2-9}{x}=\lim_{x\to 0}\frac{6x+x^2}{x}=\lim_{x\to 0}\frac{x(6+x)}{x}=\lim_{x\to 0}(6+x)$

$$=6+0=6.$$

Example 5:

$Find\ \lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$.

Solution:

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}=\lim_{x\to 0}\left[\frac{\sqrt{x^2+9}-3}{x^2}\right]\left[\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}\right]=\lim_{x\to 0}\frac{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)}{x^2(\sqrt{x^2+9}+3)}$$

$$= \lim_{x \to 0} \frac{x^2+9-9}{x^2(\sqrt{x^2+9}+3)} = \lim_{x \to 0} \frac{x^2}{x^2(\sqrt{x^2+9}+3)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{\sqrt{0+9}+3} = \frac{1}{6}.$$

Example 6:

Show that $\lim_{x \to 0} |x| = 0$.

Solution:

We define the absolute value of x as follows:

$$|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}.$$

First, we find $\lim_{x \to 0^-} |x|$.

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} = (-x) = 0.$$

Second, we find $\lim_{x \to 0^+} |x|$.

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0.$$

Since $\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} |x| = 0$.

Then $\lim_{x \to 0} |x| = 0$.

Example 7:

$$Let\ f(x) = \begin{cases} \sqrt{x-4}, & if\ x > 4 \\ 8-2x, & if\ x < 4 \end{cases}.$$

Determine whether $\lim_{x \to 4} f(x)$ exists.

Solution:

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8-2x) = 8-8 = 0.$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0.$$

Since $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = 0$.

Then $\lim_{x \to 4} f(x) - 0$.

H.W.1

Q1: Evaluate the following limits:

1: $\lim_{x \to -2}(3x^4 + 2x^2 - x + 1)$ 2: $\lim_{x \to 2} \frac{2x^2+1}{x^2+6x-4}$ 3: $\lim_{x \to 4^-} \sqrt{16 - x^2}$

4: $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$ 5: $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4}$ 6: $\lim_{x \to 4} \frac{x^2-4x}{x^2-3x-4}$ 7: $\lim_{x \to -1} \frac{x^2-4x}{x^2-3x-4}$

8: $\lim_{x \to 1} \frac{x^3-1}{x^2-1}$ 9: $\lim_{x \to 0} \frac{(2+x)^3-8}{x}$ 10: $\lim_{x \to 2} \frac{x^4-16}{x-2}$ 11: $\lim_{x \to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

12: $\lim_{x \to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$.

Q2: (1) If $1 \leq f(x) \leq x^2 + 2x + 2$, *for all* x . *Find* $\lim_{x \to -1} f(x)$.

(2) If $3x \leq f(x) \leq x^3 + 2$, *for all* x. *Find* $\lim_{x \to 1} f(x)$.

Q3: Find the following limits if exist. If not, explain why.

1: $\lim_{x \to -4} |x + 4|$ 2: $\lim_{x \to -4^-} \frac{|x+4|}{x+4}$ 3: $\lim_{x \to 2} \frac{|x-2|}{x-2}$ 4: $\lim_{x \to 1.5} \frac{2x^2-3x}{|2x-3|}$.

4: Limits of Trigonometric Functions:

Theorem 12:

$\lim_{x\to 0}\frac{\sin x}{x} = 0$, *where x mesures in radians*.

Example 8:

Find the following limits:

$1: \lim_{x\to 0}\frac{\cos x-1}{x} \quad 2: \lim_{x\to 0}\frac{\sin 2x}{5x} \quad 3: \lim_{x\to 0}\frac{\tan x \sec 2x}{3x}$.

Solution:

$$1: \lim_{x\to 0}\frac{\cos x-1}{x} = \lim_{x\to 0}\frac{-(1-\cos x)}{x} = -\lim_{x\to 0}\frac{(1-\cos x)}{x}$$

$$= -\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = -\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= -\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)} \qquad ((\cos^2 x+\sin^2 x = 1))$$

$$= -[\lim_{x\to 0}\frac{\sin x}{x}]\left[\lim_{x\to 0}\sin x\right][\lim_{x\to 0}\frac{1}{1+\cos x}]$$

$$= -(1)(0)\left(\frac{1}{2}\right) = 0\ .$$

$$2: \lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{1}{5}\lim_{x\to 0}\frac{\sin 2x}{x} = \frac{1}{5}\lim_{x\to 0}\frac{2\sin 2x}{2x} = \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} = \frac{2}{5}(1) = \frac{2}{5}\ .$$

$$3: \lim_{x\to 0}\frac{\tan x \sec 2x}{3x} = \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}\cdot\frac{1}{\cos 2x}}{3x} = \frac{1}{3}\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}\cdot\frac{1}{\cos 2x}}{x}$$

$$= \frac{1}{3}\lim_{x\to 0}\frac{\sin x\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos 2x}}{x} = \frac{1}{3}\lim_{x\to 0}[\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos 2x}]$$

$$= \frac{1}{3}[\lim_{x\to 0}\frac{\sin x}{x}]\left[[\lim_{x\to 0}\frac{1}{\cos x}\right][[\lim_{x\to 0}\frac{1}{\cos 2x}] = \frac{1}{3}(1)(1)(1) = \frac{1}{3}\ .$$

H.W. 2

Use $\lim_{x\to 0}\frac{\sin x}{x}=1$ to find the following limits:

1: $\lim_{x\to 0}\frac{\sin 3x}{4x}$ 2: $\lim_{x\to 0}\frac{\tan 2x}{x}$ 3: $\lim_{x\to 0}\frac{x\csc 2x}{\cos 5x}$ 4: $\lim_{x\to 0}\frac{1-\cos x}{\sin 2x}$

5: $\lim_{x\to 0}\frac{\sin x}{\sin 2x}$ 6: $\lim_{x\to 0}\frac{\tan 3x}{\sin 8x}$.

5: *Limits as $x\to \mp\infty$*:

Theorem 13:

1: $\lim_{x\to \infty} k=k$ 2: $\lim_{x\to -\infty} k=k$, *where k is a constant.*

3: $\lim_{x\to \infty}\frac{1}{x}=0$ 4: $\lim_{x\to -\infty}\frac{1}{x}=0$.

Example 9:

Find the following limits:

1: $\lim_{x\to \infty}\left[5+\frac{1}{x}\right]$ 2: $\lim_{x\to -\infty}\frac{\sqrt{3}\,\pi}{x^2}$.

Solution:

1: $\lim_{x\to \infty}\left[5+\frac{1}{x}\right]=\lim_{x\to \infty}5+\lim_{x\to \infty}\frac{1}{x}=5+0=5$.

2: $\lim_{x\to -\infty}\frac{\sqrt{3}\,\pi}{x^2}=\sqrt{3}\,\pi\lim_{x\to -\infty}\frac{1}{x^2}=\sqrt{3}\,\pi\left[\lim_{x\to -\infty}\frac{1}{x}\right]\left[\lim_{x\to -\infty}\frac{1}{x}\right]=\sqrt{3}\,\pi(0)(0)=0$.

Note 1:

To determine the limit of a rational function *as* $x\to \mp\infty$, we first divide the numerator and denominator by the highest power of x in the denominator.

Example 10:

Find the following limits:

$$1: \lim_{x \to \infty} \frac{5x^2+8x-3}{3x^2+2} \qquad 2: \lim_{x \to -\infty} \frac{11x+2}{2x^3-1}.$$

Solution:

$$1: \lim_{x \to \infty} \frac{5x^2+8x-3}{3x^2+2} \quad (\textit{Divide numerator and denominator by } x^2)$$

$$= \lim_{x \to \infty} \frac{\frac{5x^2}{x^2}+\frac{8x}{x^2}-\frac{3}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}} = \lim_{x \to \infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} = \frac{5+0-0}{3+0} = \frac{5}{3}.$$

$$2: \lim_{x \to -\infty} \frac{11x+2}{2x^3-1} \quad (\textit{Divide numerator and denominator by } x^3)$$

$$= \lim_{x \to -\infty} \frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}} = \lim_{x \to -\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} = \frac{0+0}{2-0} = 0.$$

Example 11:

$$\textit{Find} \lim_{x \to \infty} (x - \sqrt{x^2+16}.$$

Solution:

Here, we can multiply the numerator and denominator by the conjugate radical expression to obtain an equivalent algebraic result.

$$\text{Then} \lim_{x \to \infty} (x - \sqrt{x^2+16} = \lim_{x \to \infty} \frac{\left(x-\sqrt{x^2+16}\right)\left(x+\sqrt{x^2+16}\right)}{\left(x+\sqrt{x^2+16}\right)} = \lim_{x \to \infty} \frac{x^2-(x^2+16)}{\left(x+\sqrt{x^2+16}\right)}$$

$$= \lim_{x \to \infty} \frac{-16}{\left(x+\sqrt{x^2+16}\right)} = \lim_{x \to \infty} \frac{\frac{-16}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}} = \lim_{x \to \infty} \frac{\frac{-16}{x}}{1+\sqrt{1+\frac{16}{x^2}}}$$

$$= \frac{0}{1+\sqrt{1+0}} = 0.$$

H.W.3

Find the following limits:

1: $\lim\limits_{x \to \infty}\left(\sqrt{x+9}-\sqrt{x-4}\right)$ 2: $\lim\limits_{x \to -\infty}(\sqrt{x^2+3}-x)$

3: $\lim\limits_{x \to \infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$ 4: $\lim\limits_{x \to \infty}\frac{2x+3}{5x+7}$ 5: $\lim\limits_{x \to -\infty}\frac{2x^2+3}{5x^2+7}$

6: $\lim\limits_{x \to -\infty}\frac{x^2-4x+8}{3x^3}$ 7: $\lim\limits_{x \to -\infty}\frac{x^2-7x}{x+1}$ 8: $\lim\limits_{x \to \infty}\frac{x^4+x^3}{12x^3+128}$.

Lecture 5

Differentiation

1: Derivative of a Function:

Definition 1:

The derivative of the function $f(x)$ with respect to the variable x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad , provided\ that\ the\ limit\ exists.$$

Note 1:

The derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$.

Note 2:

If f' exists at every point in the domain $of\ f$, we call f differentiable.

Note 3:

The notations for the derivative are:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}\big(f(x)\big) = D\big(f(x)\big) = D_x f(x).$$

2: Differentiation Rules:

1: If f has the constant value $f(x) = c, then\ f'(x) = 0$.

For example:

$(i)\ If\ f(x) = 6\ , then\ f'(x) = 0$.

$(iii)\ If\ f(x) = \sqrt{3}\ , then\ f'(x) = 0$.

$(iii)\ If\ f(x) = \pi, then\ f'(x) = 0$.

2: If $f(x) = x^n, where\ n\ is\ any\ real\ number, then\ f'(x) = nx^{n-1}$.

For example:

$(i)\ If\ f(x) = x^4\ , then\ f'(x) = 4x^3$.

$(ii) If\ f(x) = x^{-7}\ , then\ f'(x) = -7x^{-8}$.

$(iii)\ If\ f(x) = x^{\frac{1}{4}}\ , then\ f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$.

3: If $u\ is\ d\ differentiable\ function\ of\ x\ and\ c\ is\ a\ constant, then$

$$\frac{d}{dx}(c\,u) = c\ \frac{du}{dx}.$$

For example:

$(i)\ If\ f(x) = 5x^8\ , then\ f'(x) = (5)(8)x^7 = 40x^7$.

$(ii)\ If\ f(x) = 2x^{-3}, then\ f'(x) = (2)(-3)x^{-4} = -6x^{-4}$.

$(iii)\ If\ f(x) = \frac{3}{5}x^5\ , then\ f'(x) = \frac{3}{5}(5)x^4 = 3x^4$.

4: If $u\ and\ v$ **are differentiable functions** $of\ x$ **, then their sum** $u + v$ **is differentiable at every point where** $u\ and\ v$ **are both differentiable, then**

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example:

$If\ f(x) = 2x^4 + 6x^3 - 2x^2 + 6x + 10\ , then$

$f'(x) = 8x^3 + 18x^2 - 4x + 6$.

5: If $u\ and\ v$ **are differentiable functions** $of\ x$ **, then their product** $u.v$ **is differentiable at every point where** $u\ and\ v$ **are both differentiable, then**

$$\frac{d}{dx}(u.v) = u\ \frac{dv}{dx} + v\frac{du}{dx}.$$

For example:

$If\ f(x) = (x^2 + 5x)(x^5 + 6x + 1)$**, then**

$f'(x) == (x^2 + 5x)(5x^4 + 6) +)\big(x^5 + 6x + 1\big)(2x + 5).$

6: If $u\ and\ v$ **are differentiable functions** $of\ x\ and\ if\ v \neq 0$ **, then the quotient** $\frac{u}{v}$ **is differentiable at every point where** $u\ and\ v$ **are both differentiable, then**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\ \frac{dv}{dx}}{v^2}.$$

For example:

$$If\ y = \frac{x^2-4}{x^3+8}\ , the\ \frac{dy}{dx} = \frac{(x^3+8)(2x)-(x^2-4)(3x^2)}{(x^3+8)^2}\ .$$

3: Second and Higher Order derivatives:

If $y = f(x)$ is a differentiable function, then the derivative $f'(x)$ is also a function. If f' is also a differentiable, then we can differentiate f' to get a new function of x *denoted by* f''. The function f'' is called the second derivative of f. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''.$$

If y'' is differentiable, its derivative $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$ is the third derivative of y with respect to x.

The names continue as you image, *with* $y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n}$, *denoting the* nth derivative of y with respect to x for any positive integer n. For example;

$If\ f(x) = x^4 - 3x^3 + 2x^2 - x + 100$, then

First derivative: $f'(x) = 4x^3 - 9x^2 + 4x - 1$

Second derivative: $f''(x) = 12x^2 - 18x + 4$

Third derivative: $f'''(x) = 24x - 18$

Forth derivative: $f^{(4)}(x) = 24$

Fifth derivative: $f^{(5)}(x) = 0$.

H.W.1

Q1: Find the first and second derivatives for the following functions:

$1: y = x^2 + x + 8$ $\quad 2: y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{4}x$ $\quad 3: y = 4 - 2x - x^{-3}$

$4: y = 6x^2 - 10x - 5x^{-2}$ $\quad 5: y = \frac{x^3+7}{x}$ $\quad 6: y = \frac{x^2+5x-1}{x^2}$.

Q2: Find the first derivative for the following functions:

$1: y = (3 - x^2)(x^3 - x + 1) \quad 2: y = (1 + x^2)(x^{\frac{3}{4}} - x^{-2}).$

4: Derivatives of Trigonometric Functions:

Let u is a function of x. $Then$ the derivatives of trigonometric functions are:

No.	Function	Derivative
1	$y = \sin u$	$\frac{dy}{dx} = (\cos u)(\frac{du}{dx})$
2	$y = \cos u$	$\frac{dy}{dx} = (-\sin u)(\frac{du}{dx})$
3	$y = \tan u$	$\frac{dy}{dx} = (\sec^2 u)(\frac{du}{dx})$
4	$y = \cot u$	$\frac{dy}{dx} = (-\csc^2 u)\ (\frac{du}{dx})$
5	$y = \sec u$	$\frac{dy}{dx} = (\sec u \tan u)(\frac{du}{dx})$
6	$y = \csc u$	$\frac{dy}{dx} = (-\csc u \cot u)\left(\frac{du}{dx}\right)$

Example 1:

Find first derivative for the following functions:

$1: y = -10x + 3\cos x \quad 2: y = x^2 \cos x \quad 3: y = \csc x - 4\sqrt{x} + 7$

$4: y = \frac{\cot x}{1+\cot x} \quad 5: y = \frac{4}{\cos x} + \frac{1}{\tan x}.$

Solution:

$1: y = -10x + 3\cos x$

$$\frac{dy}{dx} = -10 - 3\sin x.$$

$2: y = x^2 \cos x$

$$\frac{dy}{dx} = x^2(-\sin x) + (\cos x)(2x) = -x^2 \sin x + 2x\cos x.$$

$3: y = \csc x - 4\sqrt{x} + 7$

$\frac{dy}{dx} = -\csc x \cot x - 2x^{-\frac{1}{2}}.$

$4: y = \frac{\cot x}{1 + \cot x}$

$\frac{dy}{dx} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$

$5: y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4 \sec x + \cot x$

$\frac{dy}{dx} = 4 \sec x \tan x - \csc^2 x .$

H.W. 2

Find the first derivative for the following functions:

$1:\ y = \frac{3}{x} + 5 \sin x \quad 2: y = \sqrt{x}\, \sec x + 3 \quad 3: y = x^2 \cot x - \frac{1}{x^2}$

$4: y = (\sin x + \cos x) \sec x \quad 5:\ y = \frac{\cos x}{x} + \frac{x}{\cos x} \quad 6: y = x^2 \cos x - 2x \sin x.$

5: The Chain Rule:

Theorem 1:

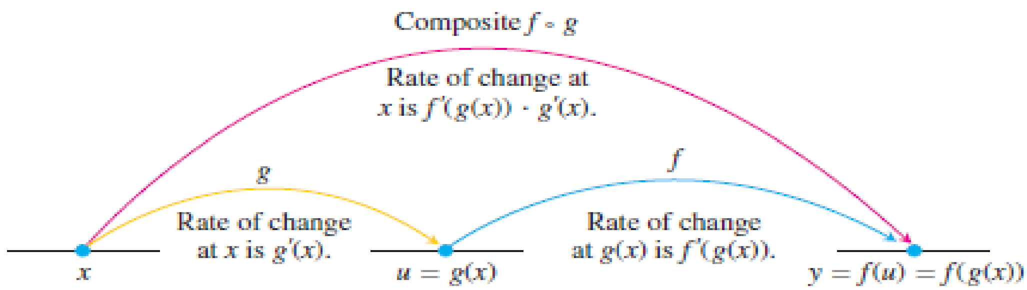
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(fog)(x) = f(g(x))$ is differentiable at x, and

$(fog)'(x) = f'(g(x))g'(x).$

Note 4:

In Leibniz's notation, if $y = f(u)$ *and* $u = g(x)$, *then* $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$,

where $\frac{dy}{du}$ *is evaluated at* $u = g(x)$.



Note 5:

The derivative of the composite function $f(g(x))$ $at\ x$ is the derivative of f at $g(x)$ times the $derivative\ of\ g\ at\ x$. This is known as the Chain Rule.

Example 2:

$Use\ Chain\ Rule\ to\ find\ \frac{dy}{dx}\ \ if\ y = sin(x^2 + 2x).$

Solution:

Let $u = x^2 + 2x.$

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (cos\ u)(2x + 2) = (2x + 2)cos(x^2 + 2x).$

Example 3:

$Use\ Chain\ Rule\ to\ find\ \frac{dy}{dx}\ \ if\ y = tan(5 - sin\ 2x).$

Solution:

Let $u = 5 - sin\ 2x$.

$Then\ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (sec^2 u)(-2cos\ 2x) = [sec^2(5 - sin\ 2x)[-2cos\ 2x].$

Note 6:

$If\ n\ is\ any\ real\ number\ and\ f\ is\ a\ power\ function, f(u) = u^n. If\ u\ is\ differentiable\ function\ of\ x, then\ \frac{d}{dx}(u^n) = nu^{n-1}\ \frac{du}{dx}.$

Example 4:

Find $\frac{dy}{dx}$ for the following functions:

$1: y = (5x^3 - x^4)^7$ $\quad 2: y = \frac{1}{3x-2}$ $\quad 3: y = sin^5 x$.

Solution:

$1: y = (5x^3 - x^4)^7$

$\frac{dy}{dx} = 7(5x^3 - x^4)^6(15x^2 - 4x^3).$

$2: y = \frac{1}{3x-2} = (3x-2)^{-1}$

$\frac{dy}{dx} = -(3x-2)^{-2}(3) = -\frac{3}{(3x-2)^2}$.

$3: y = sin^5 x$

$\frac{dy}{dx} = 5sin^4 x \, cos\, x$,

H.W. 3

Q1: Find first derivative for the following functions:

$1: y = sin(3x+1)$ $\quad 2: y = cos(sin\, x)$ $\quad 3: y = (2x+1)^5$

$4: y = (1 - \frac{x}{7})^{-7}$ $\quad 5: y = \sqrt{3x^2 - 4x + 6}$ $\quad 6: y = sin^3 x$

$7: y = x^2 sin^4 x + x cos^{-2} x$.

Q2: Find second derivative for the following functions:

$1: y = (1 + \frac{1}{x})^3$ $\quad 2: y = \frac{1}{9}\, cot(3x-1)$.

6: *L'Hopital's Rule*:

Suppose $f(a) = g(a) = 0$, such that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} .$$

Assuming that the limit on the right side of above equation exists.

Example 5:

Use L'Hopital Rule to find the following limits:

$1: \lim_{x \to 0} \frac{3x - \sin x}{x}$ $\quad 2: \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$ $\quad 3: \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

$4: \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$.

Solution:

$$1: \lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \frac{3 - 1}{1} = 2 .$$

$$2: \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2} = \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}} - \frac{1}{2}}{2x} = \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} = -\frac{1}{8} .$$

$$3: \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}}{1} = \frac{1}{2} .$$

$$4: \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} .$$

Note 7:

If $f(x) \to \mp\infty$ and $g(x) \to \mp\infty$ as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Provided that the limit on the right exists.

Example 6:

Use L'Hopital Rule to find the following limits:

$1: \lim_{x \to \frac{\pi}{2}} \frac{\sec x}{1 + \tan x}$ $\quad 2: \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution:

$$1:\ \lim_{x\to\frac{\pi}{2}} \frac{\sec x}{1+\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \frac{1}{1} = 1\ .$$

$$2:\ \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{x \sin x} = \lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{\sin x}{-x\sin x + \cos x + \cos x} = \frac{0}{2} = 0\ .$$

H.W. 4

Use L'Hopital Rule to find the following limits:

$$1:\ \lim_{x\to -5} \frac{x^2-25}{x+5} \qquad 2:\ \lim_{x\to 1} \frac{3x^3-3}{4x^3-x-3} \qquad 3:\ \lim_{x\to\infty} \frac{x-8x^2}{12x^2+5x}$$

$$4:\ \lim_{x\to 0} \frac{\sin x - x}{x^3} \qquad 5:\ \lim_{x\to 0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)\ .$$

Lecture 6

Inverse Trigonometric Functions

1: One – to – One Function:

A function $f(x)$ is said to be one-to-one (1 – 1) function on a domain D if:

$f(x_1) = f(x_2), then\ x_1 = x_2\ , for\ x_1\ and\ x_2\ in\ the\ domain\ of\ f(x).$

Example 1:

Determine whether the following functions are one-to-one or not.

$(a)\ f(x) = \sqrt{x}$ $(b)\ f(x) = x^3$ $(c)\ f(x) = x^2$.

Solution:

$(a)\ f(x) = \sqrt{x}$

Let $f(x_1) = f(x_2)$ for $x_1\ and\ x_2$ in the domain of $f(x)$.

$\rightarrow \sqrt{x_1} = \sqrt{x_2}\ \rightarrow\ x_1 = x_2\ \rightarrow f(x) = \sqrt{x}$ **is one-to-one function.**

$(b)\ f(x) = x^3$

Let $f(x_1) = f(x_2)$ for $x_1\ and\ x_2$ in the domain of $f(x)$.

$\rightarrow x_1^3 = x_2^3\ x_1 = x_2\ \rightarrow\ f(x) = x^3$ **is one-to-one function.**

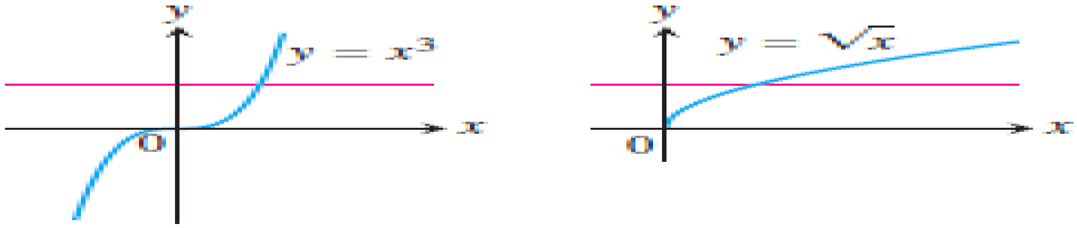
$(c)\ f(x) = x^2$

Let $f(x_1) = f(x_2)$ for $x_1\ and\ x_2$ in the domain of $f(x)$.

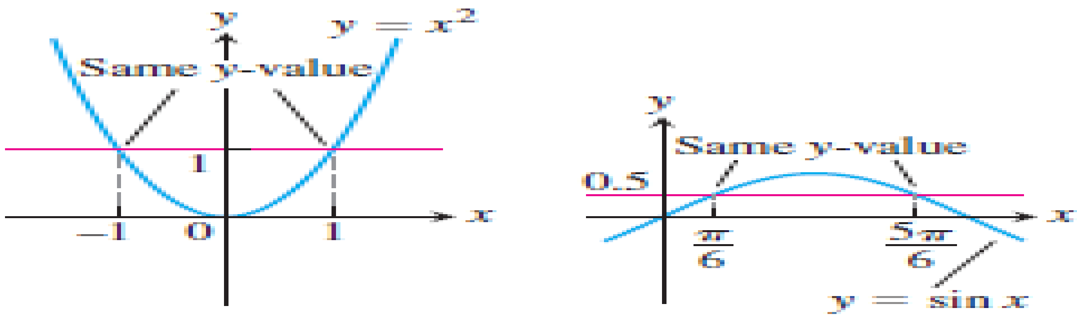
$\rightarrow x_1^2 = x_2^2\ x_1 = \mp x_2\ \rightarrow\ f(x) = x^3$ **is not one-to-one function.**

Note 1:

A function $y = f(x)$ is one-to-one function if and only if its graph intersects each horizontal line at most once.



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

2: Definition of Inverse Function:

Suppose that f is one-to-one function on a domain D *with range* R. The inverse function f^{-1} is defined by $f^{-1}(b) = a \text{ if } f(a) = b$, *where* $a \in D$ *and* $b \in R$.

The domain of f^{-1} is R and the range of f^{-1} is D.

Note 2:

1: The symbol f^{-1} for the inverse of f is read " f *inverse*".

2: $f^{-1} \neq \frac{1}{f}$.

3: $(f o f^{-1})(y) = y$, *for all* y *in the domain of* f^{-1}, *or* (*range of* f).

4: $(f^{-1} o f)(x) = x$, *for all* x *in the domain of* f.

3: How to Find Inverse Function?

The process of passing from f *to* f^{-1} can be summarized as a two-step procedure:

1: $\boldsymbol{Solve\ the\ equation\ y = f(x)\ for\ x. This\ gives\ a\ formula\ x = f^{-1}(y)}$**, where** $\boldsymbol{x}$ **is expressed as a function of** $\boldsymbol{y}$.

2: Interchange $\boldsymbol{x\ and\ y,\ obtaining\ a\ formula\ y = f^{-1}(x)}$**, where** $\boldsymbol{f^{-1}}$ **is expressed in the conventional format with** $\boldsymbol{x}$ **as the independent variable and** $\boldsymbol{y}$ **as the dependent variable.**

Example 2:

Find the inverse of $\boldsymbol{y = \frac{1}{2}x + 1}$**, expressed as a function of** $\boldsymbol{x}$.

Solution:

1: Solve for $\boldsymbol{x}$ **in terms of** $\boldsymbol{y}$.

$$y = \frac{1}{2}x + 1 \rightarrow 2y = x + 2 \rightarrow x = 2y - 2.$$

2: Interchange $\boldsymbol{x\ and\ y}$.

$$y = 2x - 2$$

Therefore $\boldsymbol{f^{-1}(x) = 2x - 2}$.

H.W. 1

Determine whether the following functions one – to –one or not. If are one – to –one find its inverses.

1: $\boldsymbol{f(x) = x^3 + 1}$ **2**: $\boldsymbol{f(x) = x^4,\ for\ x \geq 0}$ **3**: $\boldsymbol{f(x) = \frac{x+3}{x-2}}$

4: Inverse Trigonometric Functions:

The inverse trigonometric functions are denoted by:

1: $\boldsymbol{y = sin^{-1}x\quad or\ \ y = arcsin\ x}$.

2: $\boldsymbol{y = cos^{-1}x\quad or\ \ y = arccos\ x.}$

3: $\boldsymbol{y = tan^{-1}x\quad or\ \ y = arctan\ x.}$

4: $\boldsymbol{y = cot^{-1}x\quad or\ \ y = arccot\ x.}$

5: $\boldsymbol{y = \ sec^{-1}x\quad or\ y = arcsec\ x}$.

$6: y = csc^{-1}x \quad or \quad y = arccsc\ x.$

Note 3:

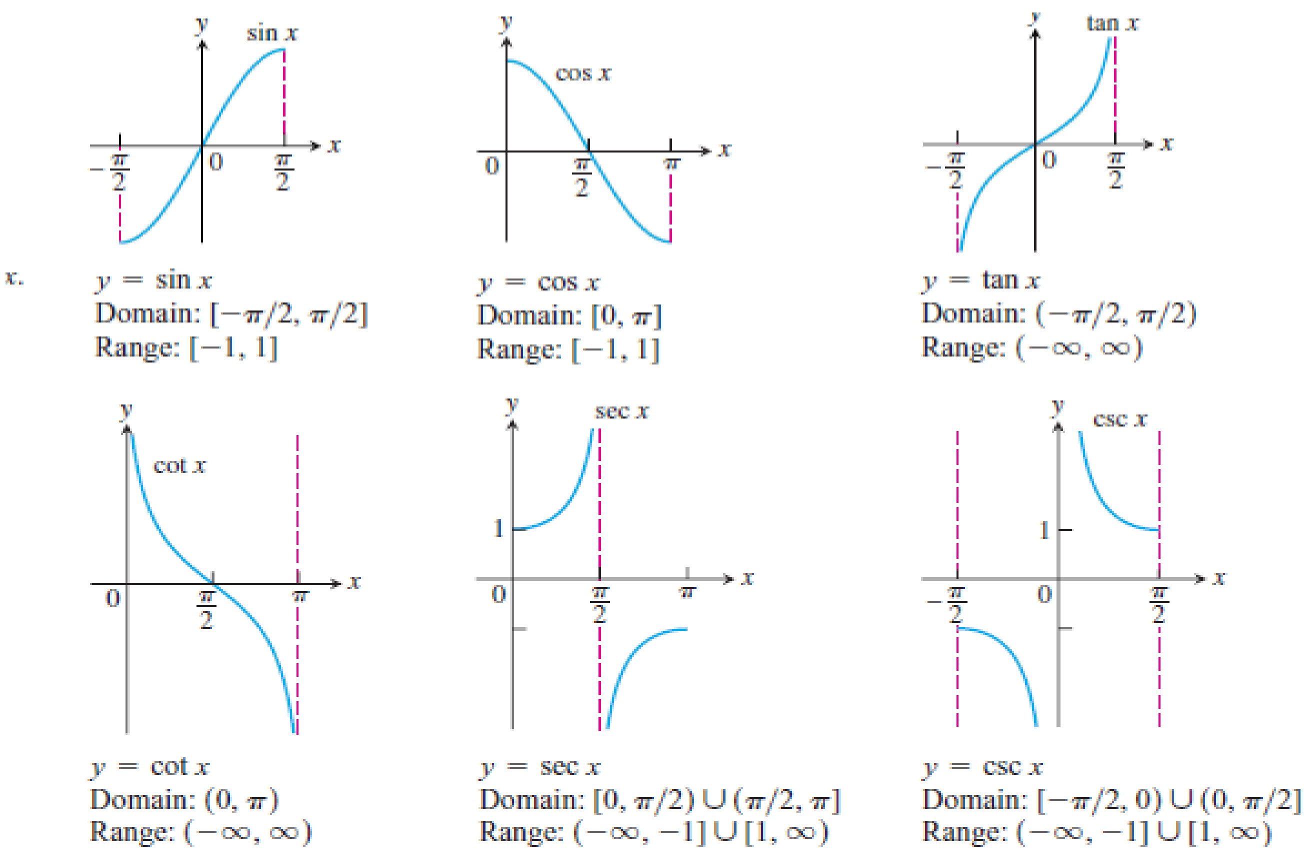
The above equations are read " $y\ equals\ arcsin\ x$**" and so on.**

Note 4:

The (-1) **in the expressions for the inverse means "** $inverse$**". It does not mean reciprocal. For example** $sin^{-1}x \neq \frac{1}{sin\ x}$,

$$where\ (sin\ x)^{-1} = \frac{1}{sin\ x} = csc\ x\ .$$

Graphs of Inverse Trigonometric Functions:



Derivatives of Inverse Trigonometric Functions:

Let u is d differential function of x. $Then$ the derivatives of the inverse trigonometric functions are given below.

No.	Function	Derivative
1	$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \lvert u \rvert < 1$
2	$y = \cos^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \lvert u \rvert < 1$
3	$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$
4	$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2} \frac{du}{dx}$
5	$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u \rvert\sqrt{u^2-1}} \frac{du}{dx}, \lvert u \rvert > 1$
6	$y = \csc^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u \rvert\sqrt{u^2-1}} \frac{du}{dx}, \lvert u \rvert > 1$

Example 3:

Find $\frac{dy}{dx}$ for the following functions:

$1: y = \sin^{-1}(5x^3)$ $2: y = \cos^{-1}(x^2)$ $3: y = \tan^{-1}(\sin x)$

$4: y \cot^{-1}(2x)$ $5: y = \sec^{-1}(\cos x)$ $6: y = \csc^{-1}(\tan x)$

Solution:

$1: y = \sin^{-1}(5x^3)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-25x^6}}(15x^2) = \frac{15x^2}{\sqrt{1-25x^6}}.$$

$2: y = \cos^{-1}(x^2)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^4}}(2x) = -\frac{2x}{\sqrt{1-x^4}}.$$

$3: y = \tan^{-1}(\sin x)$

$$\frac{dy}{dx} = \frac{1}{1+\sin^2 x}(\cos x) = \frac{\cos x}{1+\sin^2 x}.$$

4: $y = \cot^{-1} 2x$

$$\frac{dy}{dx} = -\frac{1}{1+4x^2}(2) = -\frac{2}{1+4x^2} .$$

5: $y = \sec^{-1}(\cos x)$

$$\frac{dy}{dx} = \frac{1}{|\cos x|\sqrt{\cos^2 x-1}}(-\sin x) .$$

6: $y = \csc^{-1}(\tan x)$

$$\frac{dy}{dx} = -\frac{1}{|\tan x|\sqrt{\tan^2 x-1}}(\sec^2 x) .$$

H.W. 2

Find first derivative for the following functions:

1: $y = \cos^{-1}(x^2)$ **2**: $y = \sin^{-1}(\sqrt{2x}), x > 0$ **3**: $y = \sec^{-1}(2x+1)$

4: $y = \csc^{-1}(x^2+1)$ **5**: $y = \tan^{-1}(\sec x)$ **6**: $y = \cot^{-1}(\tan x)$

7: $y = \cot^{-1}\frac{1}{x} - \tan^{-1} x$ **8**: $y = x\sin^{-1} x + \sqrt{1-x^2}$.

Lecture 7

Exponential and Logarthim Functions

1: Exponential Functions:

Definition 1:

The function $f(x) = a^x, where\ a > 0\ , a \neq 1$, is called an exponential function with base a. $For\ example$, $f(x) = 5^x\ , f(x) = (\frac{1}{2})^x$.

Rules of Exponents:

If $a > 0\ and\ b > 0\ , a \neq 1, b \neq 1, the\ following\ rules\ hold$

$for\ all\ real\ numbers\ x\ and\ y.$

$1: a^x . a^y = a^{x+y} \quad 2: \dfrac{a^x}{a^y} = a^{x-y} \quad 3: (a^x)^y = a^{xy} \quad 4: a^x . b^x = (ab)^x$

$5: \frac{a^x}{b^x} = (\frac{a}{b})^x \quad 6: a^0 = 1 \quad 7: a^{-x} = \frac{1}{a^x}$.

Definition 2:

The function $f(x) = e^x$ is called the natural exponential function, where $e\ is$ irrational number, and its value is 2.718281828 … …

2: Logarithmic Functions:

Definition 3:

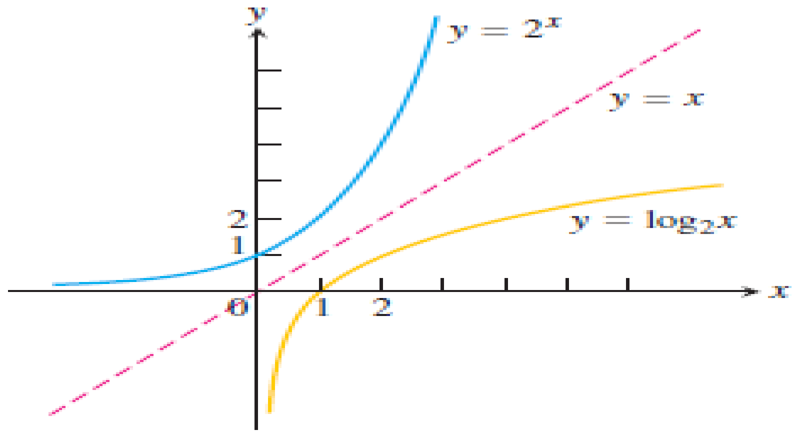
The logarithm function with base $a, y = \log_a x$, is the inverse of the exponential $function\ with\ base\ a,\ y = a^x, where\ a > 0, a \neq 1$).

Note 1:

The domain of , $y = \log_a x,$, $where\ a > 0, a \neq 1$ is the set of all positive real numbers and its range is the set of all real numbers. That is:

$Domain = \{x: x > 0\} and\ Range = \{y: y \in (-\infty, \infty)$.

Graph of the Logarithmic Functions:

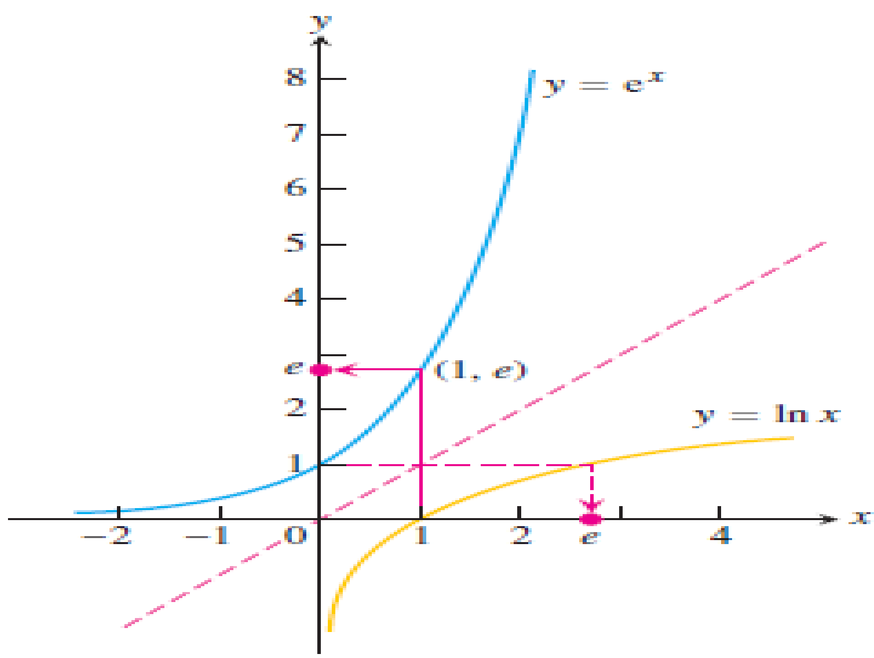


Definition 4:

The natural logarithm function is defined as $y = \ln x$, *which is* $\log_e x$.

That is: $\ln x = \log_e x$.

Graph of the Natural Logarithmic Functions:



Definition 5:

The common logarithm function is $\boldsymbol{defined\ as\ y = log\ x, whic\ is\ \log_{10} x}$**.**

That is $\log_{10} x = log\ x$ **.**

Note 2:

$1: ln\ x = y\ if\ and\ only\ if\ e^y = x\ ..\ x > 0$.

$2: ln\ e = 1 \quad 3: ln\ 1 = 0.$

Properties of the Natural Logarithm:

For any numbers $b > 0\ and\ x > 0$**, the natural logarithm satisfies the following rules:**

$1: ln(bx) = ln\ b + ln\ x \quad 2: ln\ \frac{b}{x} = ln\ b - ln\ x \quad 3: lnx^r = rln\ x$

$4: ln\frac{1}{x} = -ln\ x$.

Note 3:

$1: a^{\log_a x} = x\ \ and\ \log_a a^x = x, a > 0, a \neq 1, x > 0$.

$2: e^{ln\ x} = x\ and\ ln\ e^x = x, x > 0$.

$3: a^x = e^{x\ ln\ a}, a > 0, a \neq 1$.

$4: \log_a x = \frac{ln\ x}{ln\ a}, a > 0, a \neq 1, x > 0$.

Derivative of Exponential Functions:

Let u **is a function of** x**. Then the derivative of exponential functions are:**

No.	Function	Derivative
1	$y = a^u, a > 0, a \neq 1$	$\frac{dy}{dx} = a^u\ ln\ a\ \frac{du}{dx}$
2	$y = e^u$	$\frac{dy}{dx} = e^u \frac{du}{dx}$

Example 1:

Find the first derivative for the following functions:

$1: y = e^{x^3} \quad 2: y = e^{\cos x} \quad 3: y = 5^{-2x} \quad 4: y = 3^{\tan x}.$

Solution:

$$1: y = e^{x^3}, then\ \frac{dy}{dx} = e^{x^3}(3x^2) = 3x^2 e^{x^3}.$$

$$2: y = e^{\cos x}\ , then\ \frac{dy}{dx} = e^{\cos x}(-\sin x) = -\sin x\, e^{\cos x}.$$

$$3: y = 5^{-2x}, then\ \frac{dy}{dx} = 5^{-2x} \ln 5\,(-2) = -2\,(\ln 2)5^{-2x}.$$

$$4: y = 3^{\tan x}, then\ \frac{dy}{dx} = (3^{\tan x})(\ln 3)(\sec^2 x).$$

H.W. 1

Find the first derivatives for the following functions:

$1: y = \cos(e^x) \quad 2: y = e^{\sec 4x} \quad 3: y = x^2 e^{x^2}$

$4: y = 8^{(x^3+2x)} \qquad 5: y = 2^{-\sqrt{x}}.$

Derivative of Logarithmic Functions:

Let u is a function of x. $Then$ the derivative or the logarithmic functions are:

No.	Function	Derivative
1	$y = \ln u\ , u > 0$	$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$
2	$y = \log_a u\ , a > 0, a \neq 1, u > 0$	$\frac{dy}{dx} = \frac{1}{u}\frac{1}{\ln a}\frac{du}{dx}$

Example 2:

Find the first derivative for the following functions:

$1: y = \ln(\cos x) \quad 2: y = \ln\left(x^2 \sin(2x)\right) \quad 3: y = \log_3 e^{3x} \quad 4: y = \log(5x).$

Solution:

$$1: y = \ln(\cos x), then\ \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x.$$

2: $y = ln\left(x^2\ sin(2x)\right), then\ \frac{dy}{dx} = \frac{1}{x^2\ sin(2x)}[2x^2\ cos\ (2x) + 2x\ sin(2x)]$.

3: $y = \log_3 e^{3x}, then\ \frac{dy}{dx} = \frac{1}{e^{3x}}\frac{1}{ln\ 3}(3e^{3x})$.

4: $y = \log(5x), then\ \frac{dy}{dx} = \frac{1}{5x}\frac{1}{ln\ (10)}(5) = \frac{1}{x}\frac{1}{ln(10)}$.

H.W.2

Find the first derivative for the following functions:

1: $y = ln(sec\ x + tan\ x)$ 2: $y = ln\left(2^{x^4}\right)$ 3: $y = ln(e^{cot\ 2x})$

4: $y = \log_3(6x^2 + 8x)$ 5: $y = \log_{\sqrt{2}}\sqrt{x}$ 6: $y = \log(10^{2x})$

Lecture 8

Hyperbolic Functions and Their Inverses

1: Hyperbolic Functions

Definition of Hyperbolic Functions:

The hyperbolic functions are defined as follows:

$1:\ \sinh x = \frac{e^x - e^{-x}}{2}$ $\quad 2:\ \cosh x = \frac{e^x + e^{-x}}{2}$ $\quad 3:\ \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$4: \coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ $\quad 5: \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

$6: \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$.

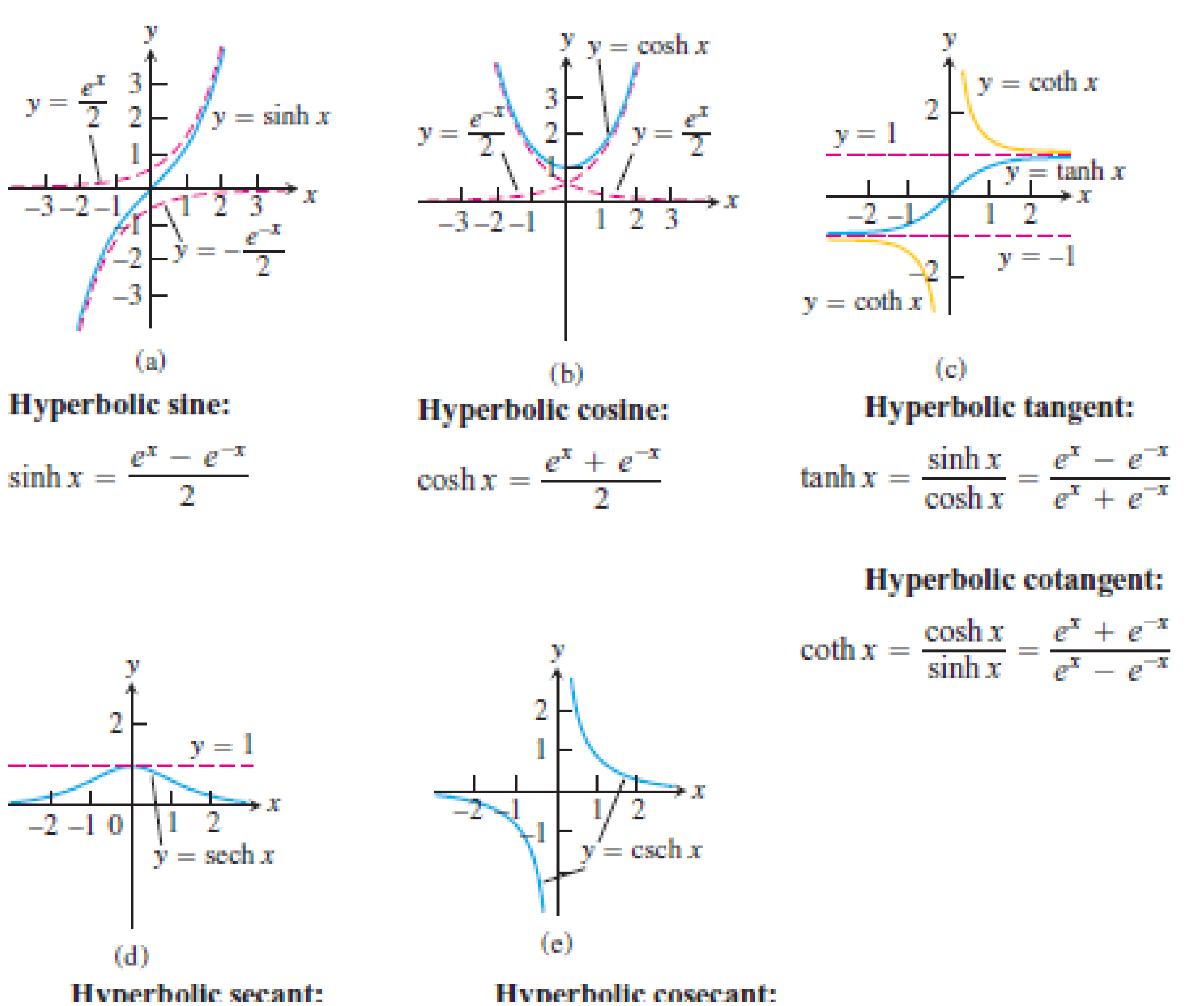
Some Identities for Hyperbolic Functions:

$1: \cosh^2 x - \sinh^2 x = 1$ $\quad 2: \sinh 2x = 2 \sinh x \cosh x$

$3: \cosh 2x = \cosh^2 x + \sinh^2 x$ $\quad 4: \cosh^2 x = \frac{\cosh 2x + 1}{2}$

$5: \sinh^2 x = \frac{\cosh 2x - 1}{2}$ $\quad 6: \tanh^2 x = 1 - \operatorname{sech}^2 x$ $\quad 7: \coth^2 x = 1 + \operatorname{csch}^2 x$.

Graphs of Hyperbolic Functions:



Derivatives of Hyperbolic Functions:

Let u is a differentiable function *with respect to* x. Then the derivatives of Hyperbolic functions are given in the following table.

No.	The Function	The Derivative
1	$y = sinh\ u$	$\frac{dy}{dx} = cosh\ u\ (\frac{du}{dx})$
2	$y = cosh\ u$	$\frac{dy}{dx} = sinh\ u\ (\frac{du}{dx})$
3	$y = tanh\ u$	$\frac{dy}{dx} = sech^2 u\ (\frac{du}{dx})$
4	$y = coth\ u$	$\frac{dy}{dx} = -csch^2 u\ (\frac{du}{dx})$

5	$y = sech\, u$	$\frac{dy}{dx} = -sech\, u\, tanh\, u\, (\frac{du}{dx})$
6	$y = csch\, u$	$\frac{dy}{dx} = -cschu\, coth\, u\, (\frac{du}{dx})$

Example 1:

Find $\frac{dy}{dx}$ for the following functions:

$1: y = 6\, sinh\, \frac{x}{3}$ $2: y = cosh(e^x)$ $3: y = tanh(ln\, x)$

$4: y = x^2 coth(x^2)$ $5: y = ln(\, sech\, x)$.

Sol.:

$1: y = 6\, sinh\, \frac{x}{3} \rightarrow \frac{dy}{dx} = 6\left(\, cosh\, \frac{x}{3}\right)\left(\frac{1}{3}\right) = 2cosh\, \frac{x}{3}$.

$2: y = cosh(e^x) \rightarrow \frac{dy}{dx} = sinh(e^x)[e^x] = e^x sinhe^x$.

$3: y = tanh(ln\, x) \rightarrow \frac{dy}{dx} = sech^2(ln\, x)\left[\frac{1}{x}\right] = \frac{1}{x} sech^2(ln\, x)\, , x > 0$.

$4: y = x^2 coth(x^2) \rightarrow \frac{dy}{dx} = x^2[-csch^2(x^2)\{2x\}] + coth(x^2)[2x]$

$= -2x^3 csch^2\, (x^2) + 2x\, coth(x^2)$.

$5: y = ln(\, sech\, x) \rightarrow \frac{dy}{dx} = \frac{1}{sech\, x}(-sech\, x\, tanh\, x) = -tanh\, x$.

2: Inverse Hyperbolic Functions

Definitions:

1: We denote the inverse of the hyperbolic sine function $by\ y = \sinh^{-1} x$ **.**

It means that for every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x .

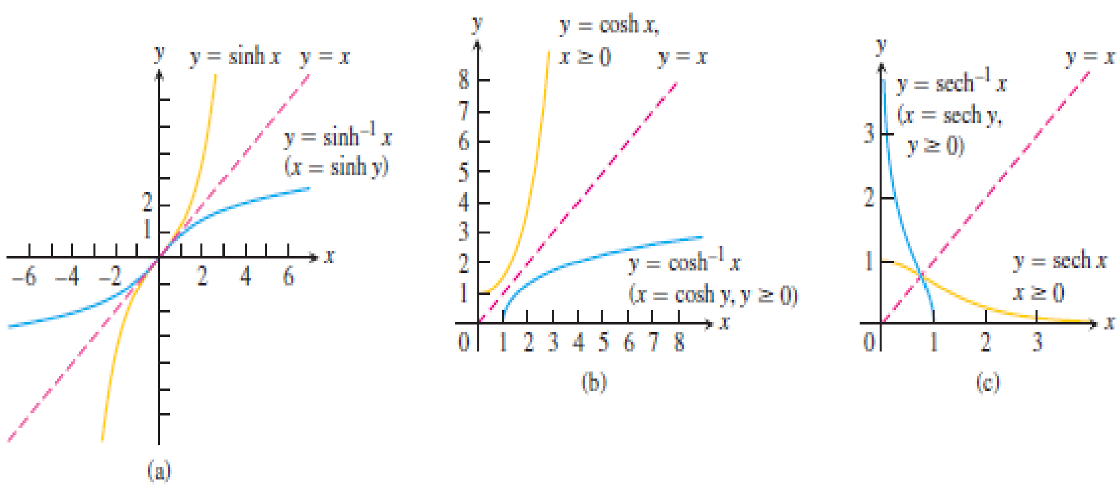
2: *The restricted function* $y = cosh\, x\, , x \geq 0$ *, is one – to – function and therefore has an inverse, is denoted by* $y = \cosh^{-1} x$ *which is the number in the interval* $0 \leq y < \infty$ *whose hyperbolic cosine is* x .

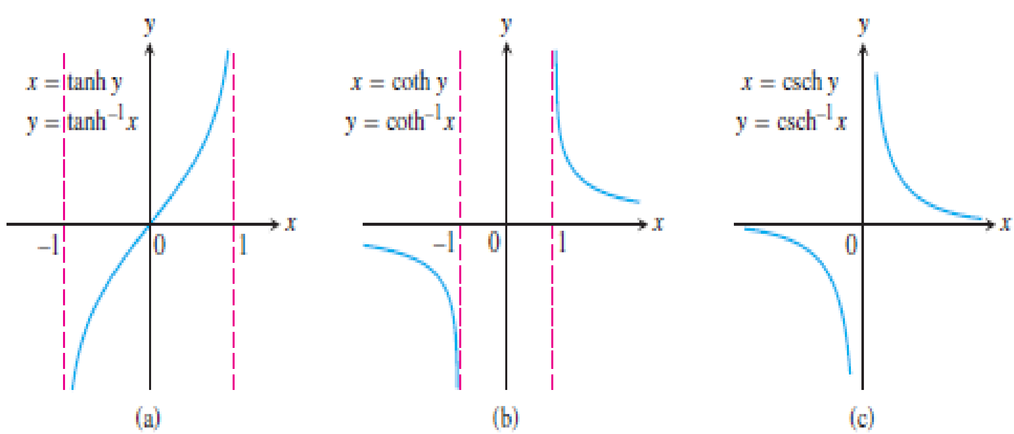
3: The restricted function $y = sech\ x$ to nonnegative values of x does have the *inverse is denoted by* $\operatorname{sech}^{-1} x$. It means that for every value of x in *the interval* $0 < x \leq 1$, $y = \operatorname{sech}^{-1} x$ is the nonnegative number whose *hyperbolic secant is* x .

4: The hyperbolic tangent, cotangent and cosecant are one – to – one functions *on* their domains and therefore have inverses are denoted by

$$y = \tanh^{-1} x, \quad y = \coth^{-1} x \;\; and \; y = \operatorname{csch}^{-1} x .$$

Graphs of Inverse Hyperbolic Functions:





Note 1:

$1:\ \operatorname{sech}^{-1} x = \cosh^{-1}\frac{1}{x}$ $\quad 2:\ \operatorname{csch}^{-1} x = \sinh^{-1}\frac{1}{x}$ $\quad 3:\ \coth^{-1} x = \tanh^{-1}\frac{1}{x}$

Derivatives of Inverse Hyperbolic Functions:

Let u is a differentiable function $of\ x$. Then the derivatives of the inverse hyperbolic functions are given in the following table.

No.	The Function	The Derivative
1	$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
2	$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx},\ u > 1$
3	$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2},\ \lvert u\rvert < 1$
4	$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2},\ \lvert u\rvert > 1$
5	$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx},\ 0 < u < 1$
6	$y = \operatorname{csch}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{1+u^2}}\frac{du}{dx},\ u \neq 0$

Example 2:

Find $\frac{dy}{dx}$ for the following functions:

$1: y = \sinh^{-1}\sqrt{x}$ $\qquad 2: y = \cosh^{-1}(sec\ x)$

Sol.:

$$1: y = \sinh^{-1}\sqrt{x} \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2}\frac{1}{\sqrt{x(1+x)}}.$$

$$2: y = \cosh^{-1}(sec\ x) \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{sec^2 x - 1}}(sec\ x\ tan\ x) = \frac{sec\ x\ tan\ x}{tan\ x}$$

$$= sec\ x\ .$$

Note 2:

We can express the inverse hyperbolic functions by logarithms as follows:

No.	Formula
1	$\sinh^{-1} x = ln\left(x + \sqrt{x^2+1}\right);\ -\infty < x < \infty$
2	$\cosh^{-1} x = ln\left(x + \sqrt{x^2-1}\right); x \geq 1$
3	$\tanh^{-1} x = \frac{1}{2}\, ln\frac{1+x}{1-x}\ ;\ \lvert x\rvert < 1$
4	$\operatorname{sech}^{-1} x = ln\left(\frac{1+\sqrt{1-x^2}}{x}\right); 0 < x \leq 1$
5	$\operatorname{csch}^{-1} x = ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right); x \neq 0$
6	$\coth^{-1} x = \frac{1}{2}\, ln\frac{x+1}{x-1}\ ;\ \lvert x\rvert > 1$

H.W.

Find the first derivative for the following functions:

$1: y = \frac{1}{2}\, sinh(2x+1)$ $\quad 2: y = 2\sqrt{x}\, tanh\sqrt{x}$ $\quad 3: y = ln(cosh\, x)$

$4: y = cschx\left(1 - ln(csch\, x)\right)$ $\quad 5: y = \cosh^{-1} 2\sqrt{x+1}$

$6: y = (x^2+2x)\tanh^{-1}(x+1)$ $\quad 7: y = \cosh^{-1} 2x - x\operatorname{sech}^{-1} 2x$

$8: y = \operatorname{csch}^{-1} 2^x$ $\quad 9: y = \sinh^{-1}(tan\, x)$ $\quad 10: y = \cosh^{-1}(sec\, x), 0 < x < \frac{\pi}{2}$

Lecture 9

Integration

1: The Definite Integral:

Let $f(x)$ be a function defined on a closed interval $[a,b]$. The definite integral is written as:

$$\int_a^b f(x)dx$$

The symbol $\int$ is an integral sign. The number a is called the lower limit of integration. The number b is called the upper limit of integration. The function $f(x)$ is called the integrand, and x is the variable of integration.

Notes:

1: $\int_a^b f(x)dx$ is read "the integral of the function f from a to b with respect to *the variable* x".

2: When you find the value of the integral, you have evaluated the integral.

3: $\int_a^b f(t)dt$ *or* $\int_a^b f(u)du$ *or* $\int_a^b f(x)dx$, it is still the same number.

Theorem (1):

If a function f is continuous over the interval $[a,b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a,b]$.

Theorem (2): (properties of Definite Integrals)

Let f & g be integrable functions over the interval $[a,b]$. The definite integral satisfies the following rules:

1: $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$.

2: $\int_a^a f(x)dx = 0$.

3: $\int_a^b k\, f(x)dx = k\int_a^b f(x)dx.$

4: $\int_a^b [f(x) \mp g(x)]dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx.$

5: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$, $where\ a < b < c$.

6: If f has maximum value M and minimum value $m\ on\ [a, b]$, then:

$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$

7: $If\ f(x) \ge g(x)\ on[a, b]$, then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.

8: $If\ f(x) \ge 0\ on[a, b], then\ \int_a^b f(x)dx \ge 0$.

Example (1):

Let $\int_{-1}^{1} f(x)dx = 5$, $\int_1^4 f(x)dx = -2\ and\ \int_{-1}^{1} h(x)dx = 7$. Evaluate:

(a) $\int_4^1 f(x)dx$ (b) $\int_{-1}^{1}[2f(x) + 3h(x)]dx$ (c) $\int_{-1}^{4} f(x)dx$

Sol.:

(a) $\displaystyle\int_4^1 f(x)dx = -\int_1^4 f(x)dx = -(-2) = 2.$

(b) $\displaystyle\int_{-1}^{1}[2f(x) + 3h(x)]dx = 2\int_{-1}^{1} f(x)dx + 3\int_{-1}^{1} h(x)dx = 2(5) + 3(7)$

$= 10 + 21 = 31$.

(c) $\displaystyle\int_{-1}^{4} f(x)dx = \int_{-1}^{1} f(x)dx + \int_1^4 f(x)dx = 5 - 2 = 3.$

Example (2):

Show that the value of $\int_0^1 \sqrt{1 + cosx}\ dx$ is less than or equal to $\sqrt{2}$.

Sol.:

From Rule 6 in Theorem 2:

$The\ lower\ bound\ for\ the\ value\ of\ the\ \int_a^b f(x)\ dx\ is\ m(b-a)\ and$

$the\ upper\ bound\ is\ M(b-a).\ Therefore, the\ maximum\ value\ of$

$\sqrt{1+cosx}$ $on\ [0,1]\ is\ \sqrt{1+1}=\sqrt{2}$.

So, $\int_0^1 \sqrt{1+cosx}\ dx \leq \sqrt{2}(1-0)=\sqrt{2}$.

Definition (1): (Area under the Curve):

Let $y=f(x)$ is nonnegative and integrable on $[a,b]$, then the area the curve $y=f(x)$ $on\ [a,b]\ is\ the\ integral\ of\ f\ from\ a$ to b, that is:

$A=\int_a^b f(x)dx.$

Definition (2): (Average Value of a Continuous Function)

If f is integrable on $[a,b]$, then its average value on $[a,b]$, also called its mean is given by:

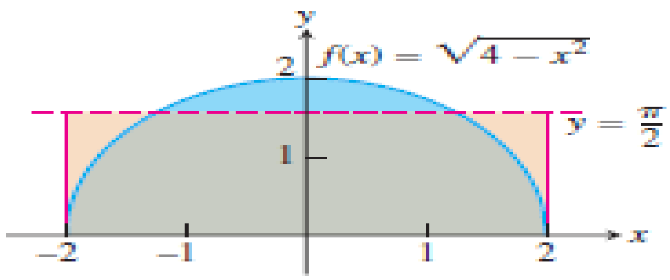
$$av(f)=\frac{1}{b-a}\int_a^b f(x)dx\ .$$

Example (3):

Find the area and the average of $f(x)=\sqrt{4-x^2}$ on $[-2,2]$.

Sol.:

The graph of the function $f(x)=\sqrt{4-x^2}$ is the upper semi-circle of radius 2 centered at the origin.



Therefore, the area is: $A=\frac{1}{2}\pi r^2=\frac{1}{2}\pi(2)^2=2\pi$.

The average value of the function is: $av=\frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\ dx=\frac{\pi}{2}$.

H.W. (1)

1: Suppose that f & g are integrable and that $\int_1^2 f(x)dx = -4$, $\int_1^5 f(x)dx = 6$, $\int_1^5 g(x)dx = 8$. Use the rules in Theorem 2 to find:

$(a)\ \int_2^2 g(x)dx \quad (b)\ \int_5^1 g(x)dx \quad (c)\ \int_1^2 3f(x)dx \quad (d)\ \int_2^5 f(x)dx$

$(e)\ \int_1^5 [f(x) - g(x)]dx \quad (i)\ \int_1^5 [4f(x) - 2g(x)]dx$.

2: *Suppose* $\int_1^2 f(x)dx = 5$. Find:

$(a)\ \int_1^2 f(u)du \quad (b)\ \int_1^2 \sqrt{3}\, f(z)dz \quad (c)\ \int_2^1 f(t)dt \quad (d)\ \int_1^2 -f(x)dx$.

3: Suppose that f is integrable and that $\int_0^3 f(z)dz = 3$ *and* $\int_0^4 f(z)dz = 7$.

***Find*: $(a)\ \int_3^4 f(z)dz \quad (b)\ \int_4^3 f(t)dt$.**

4: Use areas to evaluate the integrals:

$(a)\ \int_{-3}^{3} \sqrt{9 - x^2}\ dx \qquad (b)\ \int_{-4}^{0} \sqrt{16 - x^2}\ dx$

5: Find the average value for the functions in exercise 4.

Theorem 3: (The Mean Value Theorem for Definite Integrals)

If f is continuous on$[a,b]$, then at some point $c\ \ in\ [a,b]$,

$$f(c)=\frac{1}{b-a}\int_a^b f(x)dx\ .$$

Theorem 4-1: (The Fundamental Theorem of Calculus Part 1)

If f is continuous on$[a,b]$, then $F(x)=\int_a^x f(t)dt$ is continuous on$[a,b]$ and differentiable on(a,b), and its derivative is $f(x)$, that is:

$$F'(x)=\frac{d}{dx}\left[\int_a^x f(t)dt\ \right]=f(x).$$

Example 4:

Use Theorem 4-1 to find $\frac{dy}{dx}$.

$(a)\ y=\int_a^x (t^2+1)dt \qquad (b)\ \ y=\int_x^5 3tsintdt$

Sol.:

$(a)\ \frac{dy}{dx}=x^2+1\ .$

(b) We write the given function as

$$y=\int_x^5 3tsintdt=-\int_5^x 3tsintdt.$$

Then $\frac{dy}{dx}=-3xsinx$.

Note 1:

Let $u(x)$ & $v(x)$ are differentiable functions with respect to x and

$y=\int_{u(x)}^{v(x)} f(t)\ dt$. Then $\frac{dy}{dx}=f\big(v(x)\big)\frac{dv}{dx}-f(u(x))\frac{du}{dx}$.

Example 5:

Find $\frac{dy}{dx}$ for the following functions:

$(a)\ y=\int_1^{x^2} costdt \qquad (b)\ \ y=\int_{1+3x^2}^{4}\frac{1}{2+e^t}dt \qquad (c)\ \ y=\int_{2x}^{x^2} sintdt$

Sol.:

$(a)\ \frac{dy}{dx} = cosx^2[2x] - cos(1)[0] = 2xcosx^2 - 0 = 2xcosx^2$.

$(b)\ \frac{dy}{dx} = \frac{1}{2+e^4}(0) - \frac{1}{2+e^{(1+3x^2)}}(6x) = -\frac{6x}{2+e^{(1+3x^2)}}$.

$(c)\ \frac{dy}{dx} = sin(x^2)[2x] - sin(2x)[2] = 2xsinx^2 - 2sin2x$.

Theorem 4-2: (The Fundamental Theorem of Calculus Part 2)

If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Note 2:

Theorem 4-2 says that to calculate the definite integral of f over $[a, b]$ we need only two things:

1: Find an antiderivative $F\ of\ f$.

2: Calculate $the\ number\ F(b) - F(a)$, which is equal to $\int_a^b f(x)dx$.

Note 3:

The usual notation for the difference $F(b) - F(a)$ is $F(x)]_a^b$ or $[F(x)]_a^b$ depending on whether F has one or more terms.

Example 6:

Evaluate the following integrals:

$(a)\ \int_0^{\pi} cosx\ dx \quad (b)\ \int_0^{\frac{\pi}{4}} secx\ tanx\ dx \quad (c)\ \int_0^1 \frac{dx}{x+1}$.

Sol.:

$(a)\ \int_0^{\pi} cosx\ dx = sinx]_0^{\pi} = sin\pi - sin0 = 0 - 0 = 0$.

$(b)\ \int_0^{\frac{\pi}{4}} secx\ tanx\ dx = secx]_0^{\frac{\pi}{4}} = sec\frac{\pi}{4} - sec0 = \sqrt{2} - 1$.

$(c)\ \int_0^1 \frac{dx}{x+1} = ln(x+1)]_0^1 = ln2 - ln1 = ln2 - 0 = ln2$.

<u>Theorem 5: (The Net Change Theorem)</u>

The net change in a function $F(x)$ over $[a,b]$ is the integral of its rate of change. That $is\ F(b) - F(a) = \int_a^b F'(x)\, dx$

<u>Note 4:</u>

$F'(x)$ $represents\ the\ rate\ of\ change\ of\ the\ function\ F(x)\ with\ respect\ to\ x$, so the integral of $F'(x)$ is just the net change in $F(x)$ as x change from a to b .

<u>Example 7:</u>

If an object with position function $S(t)$ moves along a coordinate line, its velocity is $v(t) = S'(t)$. Theorem 5 $says\ that\ \int_{t_1}^{t_2} v(t)dt = S(t_2) - S(t_1)$. So the integral of velocity is the displacement over the $time\ interval$

$t_1 \le t \le t_2$.

<u>Example 8:</u>

The velocity of the rock at any time t during its motion was gives as

$v(t) = 160 - 32t\ ft/sec$.

1: Find the displacement of the rock during the time period $0 \le t \le 8$.

2: Find the total distance traveled during this time period.

<u>Sol.:</u>

1: The displacement is:

$$\int_0^8 v(t)dt = \int_0^8 (160 - 32t)dt = [160t - 16t^2]_0^8$$

$$= [160(8) - 16(64)] - [0 - 0] = 1280 - 1024 = 256\ ft .$$

2: The integral of the speed $|v(t)|$ is the total distance over the time interval.

So $$|v(t)| = \begin{cases} 160 - 32t\ if\ (160 - 32t) \ge 0 \\ 32t - 160\ if\ (160 - 32t) < 0 \end{cases}$$

$$= \begin{cases} 160 - 32t \ if \ \ 5 \geq t \\ 32t - 160 \ if \ \ 5 < t \end{cases}$$

$$= \begin{cases} 160 - 32t \ \ if \ 0 \leq t \leq 5 \\ 32t - 160 \ \ if \ 5 < t \leq 8 \end{cases} .$$

Therefore

$$\int_0^8 |v(t)| \, dt = \int_0^5 (160 - 32t)dt + \int_5^8 (32t - 160)dt$$

$$= [160t - 16t^2]_0^5 + \ [16 \, t^2 - 160t]_5^8$$

$$= [160(5) - 16(25)] + [16(64) - 160(8)] - [16(25) - 160(5)]$$

$$= [800 - 400] + [1024 - 1280] - [400 - 800]$$

$$= 400 - 256 + 400 = 800 - 256 = 544 \, ft \, .$$

Note 5:

To find the area between the graph of $y = f(x)$ and the $x - axis \ over \ [a, b]$:

1: Subdivide $[a, b]$ at the $zeros \ of \ f.$

2: Integrate f over each subinterval.

3: Add the absolute values of the integrals.

Example 9:

Find the area of the region between the $x - axis \ and \ the \ graph \ of$

$f(x) = x^3 - x^2 - 2x \, , -1 \leq x \leq 2 \, .$

Sol.:

First find the zeros of $f(x)$ as follows:

$x^3 - x^2 - 2x = 0 \ \ \rightarrow x(x^2 - x - 2) = 0 \ \rightarrow x(x + 1)(x - 2) = 0 \ \rightarrow$

$x = 0, x = -1, x = 2 \, .$

So, the zeros subdivide $[-1, 2]$ into two subintervals $[-1, 0]$ on which

$f(x) \geq 0 \ \ and \ [0, 2] \ on \ which \ f(x) \leq 0.$

Therefore,

$$A_1 = \int_{-1}^{0}(x^3 - x^2 - 2x)dx = \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{-1}^{0} = [0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = \frac{5}{12}.$$

$$A_2 = \int_{0}^{2}|(x^3 - x^2 - 2x)|\,dx = \left|\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{0}^{2}\right| = \left|\left(4 - \frac{8}{3} - 4\right) - 0\right| = \frac{8}{3}$$

Then, the total area is equal to

$$A = A_1 + A_2 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \quad square\ units\,.$$

H.W. 2

I: Evaluate the following integrals:

1: $\int_{-2}^{0}(2x + 5)dx$ 2: $\int_{0}^{2} x(x - 3)dx$ 3: $\int_{0}^{1}\left(x^2 + \sqrt{x}\right)dx$ 4: $\int_{0}^{\frac{\pi}{3}} sec^2x\,dx$

5: $\int_{0}^{\frac{\pi}{3}} 4secx\,tanx\,dx$ 6: $\int_{0}^{\frac{\pi}{4}} tan^2x\,dx$ 7: $\int_{0}^{\frac{\pi}{3}}(cosx + sec\,x)^2\,dx$

8: $\int_{1}^{2}\left(\frac{1}{x} - e^{-x}\right)dx$ 9: $\int_{2}^{4} x^{\pi-1}\,dx$.

II: Find $\frac{dy}{dx}$ for the following functions:

1: $y = \int_{0}^{x}\sqrt{1 + t^2}\,dt$ 2: $y = \int_{\sqrt{x}}^{0} sin(t^2)dt$ 3: $y = x\int_{2}^{x^2} sin(t^3)dt$

4: $y = \int_{0}^{sin^{-1}x} cost\,dt$.

III: Find the total area between the regions and the $x - axis$.

1: $y = -x^2 - 2x\,, -3 \le x \le 2$.

2: $y = x^3 - 3x^2 + 2x\,, 0 \le x \le 2$.

3: $y = \sqrt[3]{x} - x\,, -1 \le x \le 8$.

2: Indefinite Integrals and the Substitution Method

Theorem 6:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is $continuous\ on\ I$, then $\int f\big(g(x)\big)g'(x)dx = \int f(u)du$.

Example 10:

$Find \int x^3 \cos(x^4+2)dx$.

Sol.:

$Let\ u = x^4 + 2 \ \rightarrow du = 4x^3dx \ \rightarrow \ x^3dx = \frac{1}{4}du$.

So, $\int x^3 \cos(x^4+2)dx = \int \frac{1}{4} \cos u\, du = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4+2\ ) + C$.

Example 11:

Evaluate $\int \sqrt{2x+1}\ dx$.

Sol.:

First way:

Let $u = 2x+1 \ \rightarrow du = 2dx \ \rightarrow dx = \frac{1}{2}du$.

$So, \int \sqrt{2x+1}\ dx = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}\left((2x+1)^{\frac{3}{2}}\right) + C.$

Second way:

Let $u = \sqrt{2x+1} \ \rightarrow du = \frac{dx}{\sqrt{2x+1}} \ \rightarrow dx = \sqrt{2x+1}\, du = udu$.

$So, \int \sqrt{2x+1}\ dx = \int u\, udu = \int u^2du = \frac{1}{3}u^3 + C = \frac{1}{3}(\sqrt{2x+1})^3 + C$.

Example 12:

Find $\int \frac{xdx}{\sqrt{1-4x^2}}$.

Sol.:

$Let\ u = 1 - 4x^2 \ \rightarrow du = -8xdx \ \rightarrow xdx = -\frac{1}{8}du.$

$$So, \int \frac{xdx}{\sqrt{1-4x^2}} = \int \frac{1}{\sqrt{u}} \left(-\frac{1}{8}du\right) = -\frac{1}{8}\int u^{-\frac{1}{2}}\, du = -\frac{1}{8}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= -\frac{1}{4}\left((1-4x^2)^{\frac{1}{2}}\right) + C = -\frac{1}{4}\sqrt{1-4x^2} + C \,.$$

<u>Example 13:</u>

$Calculate\ \int e^{5x}\, dx$ **.**

<u>Sol.:</u>

$$Let\ u = 5x \rightarrow du = 5dx \rightarrow \frac{1}{5}du = dx \,.$$

$$So, \int e^{5x}\, dx = \int e^u \frac{1}{5} du = \frac{1}{5}e^u + C = \frac{1}{5}\, e^{5x} + C \,.$$

<u>Example 14:</u>

Find $\int \sqrt{1+x^2}\ x^5\, dx$.

<u>Sol.:</u>

$$Let\ u = 1 + x^2 \rightarrow du = 2xdx \rightarrow \frac{1}{2}\, du = xdx \,.$$

$$So, \int \sqrt{1+x^2}\ x^5\, dx = \int \sqrt{1+x^2}\ x^4\, x\, dx = \int \sqrt{u}\, ((u-1)^2)\frac{1}{2}\, du$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}(u^2 - 2u + 1)du = \frac{1}{2}\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{7}{2}}}{\frac{7}{2}} - 2\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right) + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + C = \frac{1}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{3}u^{\frac{3}{2}} + C$$

$$= \frac{1}{7}\left((1+x^2)^{\frac{7}{2}}\right) - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C \,.$$

<u>Example 15:</u>

Calculate $\int \tan x\, dx$.

<u>Sol.:</u>

We can write the above integral as

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$Let\ u = cosx \ \rightarrow du = -sinx\, dx \ \rightarrow \ -du = sinx\, dx$.

$So, \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -ln|u| + C = -ln|cos| + C.$

$\underline{Note\ 6:\ We\ can\ also\ write\ \int tanx\, dx = \ ln\,|secx| + C.}$

Example 16:

Sol.:

$Find\ \int \frac{dx}{e^x + e^{-x}}$.

Sol.:

$$Let\ u = e^x \ \rightarrow du = e^x dx \rightarrow e^{-x} du = dx \ \rightarrow \frac{1}{e^x} du = dx \ \rightarrow \frac{1}{u} du = dx.$$

$$So, \int \frac{dx}{e^x + e^{-x}} = \int \frac{dx}{e^x + \frac{1}{e^x}} = \int \frac{\frac{du}{u}}{u + \frac{1}{u}} = \int \frac{\frac{du}{u}}{\frac{u^2+1}{u}} = \int \frac{du}{u^2 + 1}$$

$$= \tan^{-1} u + C = \tan^{-1} e^x + C .$$

3: The integrals of $sin^2 x$ and $cos^2 x$

$$A: \int sin^2 x \, dx = \int \frac{1 - cos2x}{2} \, dx = \frac{1}{2} \int (1 - cos2x) dx$$

$$= \frac{1}{2} \int dx - \frac{1}{2} \int cos2x \, dx = \frac{1}{2} x - \frac{1}{4} \, sin2x + C .$$

$$B: \int cos^2 x \, dx = \int \frac{1 + cos2x}{2} \, dx = \frac{1}{2} \int (1 + cos2x) dx$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int cos2x \, dx = \frac{1}{2} x + \frac{1}{4} \, sin2x + C .$$

H.W. 3

I: Evaluate the following integrals by using the given substitutions to reduce the integral to standard form.

1: $\int 2(2x+4)^5\,dx\,, u = 2x+4$ 2: $\int 7\sqrt{7x-1}\,dx\,, u = 7x-1$

3: $\int 2x\,(x^2+5)^{-4}\,dx\,, u = x^2+5$ 4: $\int \frac{4x^3}{(x^4+1)^2}\,dx\,, u = x^4+1$

5: $\int x\,sin(2x^2)dx\,, u = 2x^2$ 6: $\int (1 - cos\left(\frac{x}{2}\right))\,dx\,, u = \frac{x}{2}$.

7: $\int csc^2 2x\,cot 2x\,dx\,,\ (a)\ u = cot2x\ ,\ (b)\ u = csc2x$

8: $\int \frac{dx}{\sqrt{5x+8}}\ ,\ (a)\ u = 5x+8\ \ (b)\ u = \sqrt{5x+8}$

II: Evaluate the following integrals:

1: $\int \sqrt{3-2x}\,dx$ 2: $\int 3x\sqrt{7-3x^2}\,dx$ 3: $\int sec^2\,(3x+2)\,dx$

4: $\int tan^2 x\,sec^2 x\,dx$ 5: $\int sin^5\frac{x}{3}\,cos\frac{x}{3}\,dx$ 6: $\int x^{\frac{1}{2}}\ sin\left(x^{\frac{3}{2}}+1\right)dx$

7: $\int cos\,x\,e^{sinx}\,dx$ 8: $\int \frac{dx}{x\,ln\,x}$.

4: Substitutions in Definite Integrals

Theorem 7: (Substitution in Definite Integrals)

If g' is continuous on the interval $[a,b]$ and f is continuous on the $range\ of\ g(x) = u$, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Ex. 17:

$Evaluate\ \int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Sol.:

$Let\ u = x^3 + 1 \ \rightarrow du = 3x^2\,dx$.

$When\ x = -1 \ \rightarrow u = 0\ .When\ x = 1 \ \rightarrow u = 2$.

$$So, \int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \int_0^2 u^{\frac{1}{2}}\,du = [\frac{u^{\frac{3}{2}}}{\frac{3}{2}}]_0^2 = \frac{2}{3}[2^{\frac{3}{2}} - 0]$$

$$= \frac{2}{3}[2\sqrt{2}] = \frac{4}{3}\sqrt{2}\ .$$

Example 18:

Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cot\ x\ csc^2x\ dx$.

Sol.:

$Let\ u = cot\ x \ \rightarrow du = \ -csc^2x\ dx \ \rightarrow \ -du - csc^2x\ dx$.

$When\ x = \frac{\pi}{4} \ \rightarrow u = cot\frac{\pi}{4} = 1\ .When\ x = \frac{\pi}{2} \ \rightarrow u = cot\frac{\pi}{2} = 0$.

$$So, \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cot\ x\ csc^2x\ dx = \int_1^0 -udu = \int_0^1 udu = [\frac{u^2}{2}]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}\ .$$

Example 19:

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\, x\, dx$.

Sol.:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\, x\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{sin\, x}{cos\, x}\, dx.$$

$Let\ u = cosx \rightarrow du = -sinx\, dx \rightarrow -du = sinxdx$.

$When\ x = -\frac{\pi}{4} \rightarrow u = cos\left(-\frac{\pi}{4}\right) = cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$When\ x = \frac{\pi}{4} \rightarrow u = cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$$So, \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\, x\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{sin\, x}{cos\, x}\, dx = \int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} -\frac{du}{u} = 0 .$$

Note 7:

The interval of the type $[-a, a]$ for any real number a is called a symmetric interval, for example: $[-2, 2], [-\pi, \pi], [-5, 5]$, ...

Theorem 8:

Let f be continuous on the symmetric interval $[-a, a]$.

1: If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

2: If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Example 20:

Verify Theorem 8 for the following functions:

$1: f(x) = x^4 - 4x^2 + 6\ , where\ -2 \leq x \leq 2$.

$2: f(x) = sinx\ , where\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Sol.:

1: $f(x) = x^4 - 4x^2 + 6$ **is even function.**

$$\int_{-2}^{2} f(x)dx = \int_{-2}^{2}(x^4 - 4x^2 + 6)dx = [\tfrac{1}{5}x^5 - \tfrac{4}{3}x^3 + 6x]_{-2}^{2}$$

$$= \left[\tfrac{1}{5}(2)^5 - \tfrac{4}{3}(2)^3 + 6(2)\right] - [\tfrac{1}{5}(-2)^5 - \tfrac{4}{3}(-2)^3 + 6(-2)]$$

$$= \left[\tfrac{32}{5} - \tfrac{32}{3} + 12\right] - \left[-\tfrac{32}{5} + \tfrac{32}{3} - 12\right] = \tfrac{64}{5} - \tfrac{64}{3} + 24$$

$$= \tfrac{192}{15} - \tfrac{320}{15} + \tfrac{360}{15} = \tfrac{232}{15}\,.$$

$$\int_{0}^{2} f(x)dx = \int_{0}^{2}(x^4 - 4x^2 + 6)dx = [\tfrac{1}{5}x^5 - \tfrac{4}{3}x^3 + 6x]_{0}^{2}$$

$$= \left[\tfrac{1}{5}(2)^5 - \tfrac{4}{3}(2)^3 + 6(2)\right] - 0 = \left[\tfrac{32}{5} - \tfrac{32}{3} + 12\right]$$

$$= \tfrac{96}{15} - \tfrac{160}{15} + \tfrac{180}{15} = \tfrac{116}{15}\,.$$

$Therefore,\ 2\int_{0}^{2} f(x)dx = 2(\tfrac{116}{15}) = \tfrac{232}{15}$ **.**

Thus: $\int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)\,dx$ **.**

2: $f(x) = sinx$ **is an odd function.**

$$So, \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx = [-cosx]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\left[cos\left(\tfrac{\pi}{2}\right) - cos\left(-\tfrac{\pi}{2}\right)\right]$$

$$= -\left[\, cos\left(\tfrac{\pi}{2}\right) - cos\left(\tfrac{\pi}{2}\right)\right] = 0\,.$$

5: Area between Two Curves

Let $f\ and\ g$ **are continuous functions with** $f(x) \geq g(x)\ throughout\ [a, b]$**, then the area of the region between the curves** $y = f(x)\ and\ y = g(x)$ **from** $a\ to\ b$ **is the integral of** $f - g$ **from** $a\ to\ b$**. That is:**

$$A = \int_{a}^{b}[f(x) - g(x)]dx$$

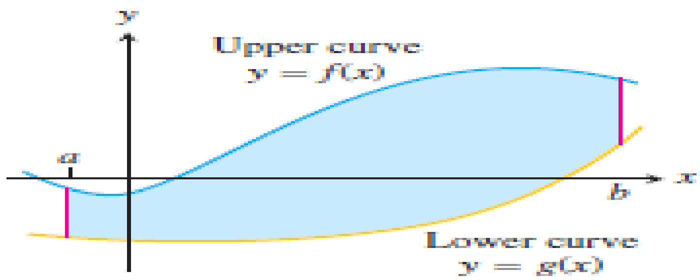


FIGURE 5.25 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

Example 21:

Find the area of the region bounded above by the $curve\ y = 2e^{-x} + x$, below by the $curve\ y = \frac{1}{2}e^{x}$, on the left by $x = 0$, and on the right by $x = 1$.

Sol.:

$$A = \int_0^1 (\ 2e^{-x} + x - \frac{1}{2}e^{x})\ dx = [-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^{x}]_0^1$$

$$= \left[-2e^{-1} + \frac{1}{2} - \frac{1}{2}e\right] - \left[-2 + 0 - \frac{1}{2}\right] = -\frac{2}{e} - \frac{e}{2} + \frac{1}{2} + \frac{1}{2} + 2$$

$$= -\frac{4}{2e} - \frac{e^2}{2e} + \frac{6e}{2e} = \frac{-4-e^2+6e}{2e}\ \ square\ units.$$

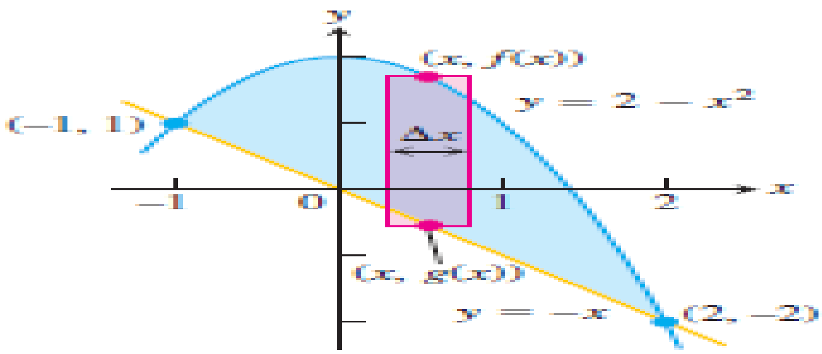
Example 22:

Find the area of the region enclosed by the Parabola $y = 2 - x^2$ and the line $y = -x$.

Sol.:

First sketch the two curves and determine the limits of integration by solving

$y = 2 - x^2\ \ and\ y = -x$.



Let $2 - x^2 = -x \;\rightarrow\; x^2 - x - 2 = 0 \;\rightarrow (x+1)(x-2) = 0 \;\rightarrow x = -1, x = 2.$

Then $A = \int_{-1}^{2} [\,2 - x^2 + x]dx = \left[2x - \frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{2}$

$$= \left[4 - \frac{8}{3} + 2\right] - \left[-2 + \frac{1}{3} - \frac{1}{2}\right] = 8 - 3 - \frac{1}{2} = \frac{9}{2} \;\; \textit{square units}.$$

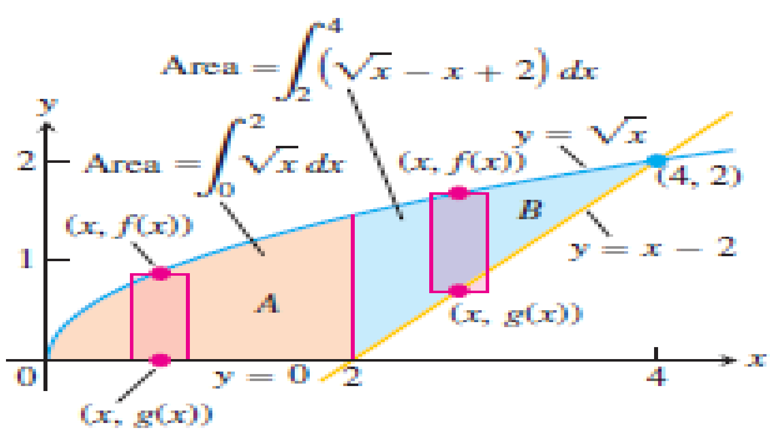
<u>Example 23:</u>

Find the area of the region in the first quadrant that is bounded above by

$y = \sqrt{x}$ ***and below by the*** $x -$ ***axis and the line*** $y = x - 2.$

<u>Sol.:</u>

First sketch the region and determine the limits of integration.



The limits of integration for the region A are from $x = 0$ ***to*** $x = 2$ **.**

The left-hand limit for the region B is $x = 2$ **.**

To find the right – hand limit for the region B, we solve the equations

$y = \sqrt{x}$ *and* $y = x - 2$.

Then $\sqrt{x} = x - 2 \rightarrow x = x^2 - 4x + 4 \rightarrow x^2 - 5x + 4 = 0$

$\rightarrow (x-1)(x-4) = 0$, $x = 1, x = 4$.

Here, only $x = 4$ *satisfies the equation* $\sqrt{x} = x - 2$.

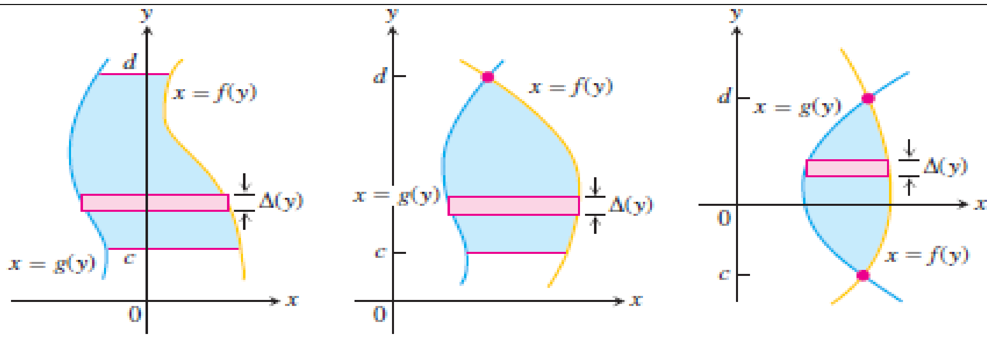
Then the right – hand limit for the region B is $x = 4$.

$$The\ total\ area$$
$$= The\ area\ of\ the\ region\ A + The\ area\ of\ the\ region\ B.$$
$$= \int_0^2 \sqrt{x}\, dx + \int_2^4 (\sqrt{x} - x + 2) dx$$
$$= \int_0^2 x^{\frac{1}{2}}\, dx + \int_2^4 \left(x^{\frac{1}{2}} - x + 2\right) dx$$
$$= [\frac{x^{\frac{3}{2}}}{\frac{3}{2}}]_0^2 + [\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^2}{2} + 2x]_2^4$$
$$= \left[\frac{2}{3}(2^{\frac{3}{2}}) - 0\right] + \left[\frac{2}{3}(4)^{\frac{3}{2}} - 8 + 8\right] - \left[\frac{2}{3}(2^{\frac{3}{2}}) - 2 + 4\right]$$
$$= \frac{2}{3}(2^{\frac{3}{2}}) - \frac{2}{3}(2^{\frac{3}{2}}) + \frac{2}{3}(4)^{\frac{3}{2}} - 8 + 8 + 2 - 4$$
$$= \frac{16}{3} - 2 = \frac{16}{3} - \frac{6}{3} = \frac{10}{3}\ \ square\ units\ .$$

Note 8:

If a bounding of regions of curves are described *by functions of y*, **then the total area is given by the following formula:**

$$A = \int_c^d [f(y) - g(y)]\, dy\ .$$



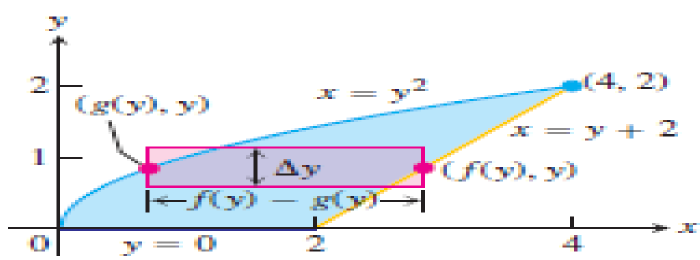
In above equation f always denotes the right – hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

<u>Example 24:</u>

Find the area of the region in the first quadrant that is bounded above by

$y = \sqrt{x}$ *and below by the* $x -$ *axis and the line* $y = x - 2$ *by integrating with respect to* y.

<u>Sol.:</u>



The right boundary is the line $x = y + 2$, *so* $f(y) = y + 2$.

The left boundary is the curve $x = y^2$, *so* $g(y) = y^2$.

The lower limit of integration $y = 0$.

We find the upper limit of integration by solving $x = y + 2$ *and* $x = y^2$ *for* y.

So, $y + 2 = y^2 \ \rightarrow \ y^2 - y - 2 = 0 \ \rightarrow (y + 1)(y - 2) = 0 \ \rightarrow y = -1, y = 2.$

Thus the upper limit of integration *is* $y = 2$.

Then the area is

$$A = \int_0^2 [y + 2 - y^2]dy = [\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3]_0^2 = \left[2 + 4 - \frac{8}{3} - 0\right]$$

$$= 6 - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \frac{10}{3} \ \ square\ units.$$

H.W. 4

I: Evaluate the following integrals:

1: $\int_0^1 \sqrt{y+1}\ \ dy$ 2: $\int_0^1 x\sqrt{1-x^2}\,dx$ 3: $\int_0^{\frac{\pi}{4}} tanx\, sec^2 x\, dx$

4: $\int_{2\pi}^{3\pi} 3\, cos^2 x\, sinx\, dx$ 5: $\int_{-1}^{1} t^3(1+t^4)^3\, dt$ 6: $\int_0^1 \frac{5rdr}{(4+r^2)^2}$

7: $\int_0^{\sqrt{3}} \frac{4x\, dx}{\sqrt{x^2+1}}$ 8: $\int_{-\frac{\pi}{2}}^{0} \left(2 + tan\frac{x}{2}\right) sec^2 \frac{x}{2} dx$ 9: $\int_0^{\pi} \frac{sint\, dt}{2-cost}$

10: $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + e^{cot\,x}) csc^2 x\, dx$ 11: $\int_0^{ln\sqrt{3}} \frac{e^x\, dx}{1+e^{2x}}$.

II: Find the area of the region enclosed by the lines and curves in the following

1: $y = x^2 - 2\ and\ y = 2$ 2: $y = x^2\ \ and\ y = \ -x^2 + 4x$

3: $y = x^2 - 2x\ and\ y = x$ 4: $y = 7 - 2x^2\ and\ y = x^2 + 4$

5: $4x^2 + y = 4\ and\ x^4 - y = 1$ 6: $x + y^2 = 3\ and\ 4x + y^2 = 0$

7: $y = 2sinx\ and\ y = sin2x\ \ , 0 \le x \le \pi$

8: $y = 8cosx\ and\ y = sec^2 x\ \ , -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

Lecture 10

Integrals and Transcendental Functions

1: $The\ Integral\ of\ \int \frac{1}{u}\,du$

Let u is a differentiable function that is never zero, then

$\int \frac{1}{u}\,du = ln|u| + C$.

Ex. 1:

$Evaluate\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4cosx}{3+2sinx}\,dx$.

Sol.:

$Let\ u = 3 + 2sinx \ \rightarrow du = 2cosx\,dx \ \rightarrow 2du = 4cosx\,dx.$

$When\ x = -\frac{\pi}{2} \ \rightarrow u = 3 + 2sin\left(-\frac{\pi}{2}\right) = 3 - 2sin\left(\frac{\pi}{2}\right) = 3 - 2(1) = 1$.

$When\ x = \frac{\pi}{2} \ \rightarrow u = 3 + 2sin\left(\frac{\pi}{2}\right) = 3 + 2(1) = 3 + 2 = 5$.

$$So, \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4cosx}{3+2sinx}\,dx = \int_1^5 \frac{2\,du}{u} = 2\int_1^5 \frac{du}{u} = 2[ln\,u]_1^5 = 2[ln5 - ln1]$$

$$= 2[ln5 - 0] = 2ln\,5\,.$$

2: Integrals of $tan\ x\ , cot\ x\ , sec\ x, csc\ x$

1: $\int tan\ x\,dx = \int \frac{sin\ x}{cos\ x}\,dx = -ln|cos\ x| + C = ln|sec\ x| + C$.

2: $\int cot\ x\,dx = \int \frac{cos\ x}{sin\ x}\,dx = ln|sin\ x| + C$.

3: $\int sec\ x\,dx = \int \frac{sec\ x(sec\ x+tan\ x)}{sec\ x+tan\ x}\,dx = \int \frac{sec^2x\ sec\ x\ tan\ x}{sec\ x+tan\ x}\,dx$

$= ln|sec\ x + tan\ x| + C$.

4: $\int csc\ x\,dx = \int \frac{csc\ x\ (\ csc\ x+cot\ x)}{csc\ x+cot\ x}\,dx = \int \frac{csc^2\ x\ +csc\ x\ cot\ x}{csc\ x+cot\ x}\,dx$

$$= -\int \frac{-\csc^2 x - \csc x \cot x}{\csc x + \cot x}\, dx = -\ln|\csc x + \cot x| + C.$$

3: The Integrals of Exponential Functions:

Let u is a differentiable function.

1: $\int e^u\, du = e^u + C$.

2: $\int a^u\, du = \dfrac{a^u}{\ln a} + C\,, a > 0\,, a \neq 1$.

Ex. 2:

Find the following integrals:

1: $\int e^{\sin x} \cos x\, dx$ 2: $\int_{\ln 2}^{\ln 3} e^x\, dx$ 3: $\int_0^1 2^{-x}\, dx$ 4: $\int_0^{\frac{\pi}{4}} (\frac{1}{3})^{\tan x} \sec^2 x\, dx$.

Sol.:

1: $\int e^{\sin x} \cos x\, dx$

$Let\ u = \sin x \rightarrow du = \cos x\, dx$.

$So, \int e^{\sin x} \cos x\, dx = \int e^u\, du = e^u + C = e^{\sin x} + C$.

2: $\int_{\ln 2}^{\ln 3} e^x\, dx = [e^x]_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$.

3: $\int_0^1 2^{-x}\, dx$

$Let\ u = -x \rightarrow du = -dx \rightarrow -du = dx$.

$When\ x = 0 \rightarrow u = 0\ . When\ x = 1 \rightarrow u = -1$.

$So, \int_0^1 2^{-x}\, dx = \int_0^{-1} 2^u\, (-du) = -\int_0^{-1} 2^u\, du = \int_{-1}^0 2^u\, du$

$$= [\frac{2^u}{\ln 2}]_{-1}^0 = \frac{2^0}{\ln 2} - \frac{2^{-1}}{\ln 2} = \frac{1}{\ln 2} - \frac{1}{2\ln 2} = \frac{1}{\ln 2}\left[1 - \frac{1}{2}\right] = \frac{1}{2\ln 2}\,.$$

4: $\int_0^{\frac{\pi}{4}} (\frac{1}{3})^{\tan x} \sec^2 x\, dx$

$Let\ u = \tan x \rightarrow du = \sec^2 x\, dx$.

$$When\ x = 0 \ \rightarrow u = tan\ 0 = 0\ .When\ x = \frac{\pi}{4} \ \rightarrow u = tan\frac{\pi}{4} = 1\ .$$

$$So, \int_0^{\frac{\pi}{4}} (\tfrac{1}{3})^{tan\,x} sec^2 x\, dx = \int_0^1 (\tfrac{1}{3})^u\, du = [\frac{(\frac{1}{3})^u}{ln\frac{1}{3}}]_0^1 = \frac{\frac{1}{3}}{ln\frac{1}{3}} - \frac{(\frac{1}{3})^0}{ln\frac{1}{3}}$$

$$= \frac{1}{ln\frac{1}{3}}\left[\frac{1}{3} - 1\right] = \frac{-\frac{2}{3}}{ln\frac{1}{3}} = \frac{-\frac{2}{3}}{ln1 - ln3} = \frac{-\frac{2}{3}}{0 - ln\,3} = \frac{\frac{2}{3}}{ln\,3} = \frac{2}{3\,ln\,3}\,.$$

H.W. 1

Evaluate the following integrals:

1: $\int_{-3}^{-2} \frac{dx}{x}$ 2: $\int \frac{2y\,dy}{y^2-25}$ 3: $\int \frac{3\,sec^2 t\,dt}{6+3\,tan\,t}$ 4: $\int 8\,e^{x+1}\,dx$ 5: $\int_1^4 \frac{(ln\,x)^3}{2x}\,dx$

6: $\int \frac{ln(ln\,x)}{x\,ln\,x}\,dx$ 7: $\int 2\,t\,e^{-t^2}\,dt$ 8: $\int_1^{\sqrt{2}} x\,2^{x^2}\,dx$ 9: $\int_1^4 \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

10: $\int_0^{\frac{\pi}{2}} 7^{cos\,t}\,sin\,t\,dt$ 11: $\int_1^2 \frac{2^{ln\,x}}{x}\,dx$ 12: $\int_1^4 \frac{\log_2 x}{x}\,d$

13: $\int_1^4 \frac{ln\,2\,\log_2 x}{x}\,dx$ 14: $\int \frac{dx}{x\,\log x}$ 15: $\int \frac{dx}{x\,(\log x)^2}$.

4: Integrals Formulas for Hyperbolic Functions:

Let u is a differentiable function.

1: $\int sinh\,u\,du = cosh\,u + C$

2: $\int cosh\,u\,du = sinh\,u + C$

3: $\int sech^2 u\,du = tanh\,u + C$

4: $\int csch^2\,u\,du = -coth\,u + C$

5: $\int sech\,u\,tanh\,u\,du = -sech\,u + C$

6: $\int csch\,u\,coth\,u\,du = -csch\,u + C$

$7: \int \tanh u\, du = \int \frac{\sinh u}{\cosh u}\, du = \ln(\cosh u) + C$

$8: \int \coth u\, du = \int \frac{\cosh u}{\sinh u}\, du = \ln|\sinh u| + C$

Ex. 4:

Evaluate the following integrals:

$1: \int \coth 5x\, dx \quad 2: \int_0^1 \sinh^2 x\, dx \quad 3: \int_0^{\ln 2} 4 \quad \sinh x\, dx$

Sol.:

$1: \int \coth 5x\, dx$

$Let\ u = 5x \to du = 5dx \to \frac{1}{5}\, du = dx$.

$So, \int \coth 5x\, dx = \int \coth u\, (\frac{1}{5}\, du) = \frac{1}{5} \int \coth u\, du = \frac{1}{5} \ln|\sinh u| + C$

$= \frac{1}{5} \ln|\sinh 5x| + C$.

$2: \int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2}\, dx = \frac{1}{2} [\int_0^1 \cosh 2x\, dx - \int_0^1 dx\,]$

$= \frac{1}{2} [\frac{\sinh 2x}{2} - x]_0^1 = \frac{1}{2} [\left(\frac{\sinh 2}{2} - 1\right) - \left(\frac{\sinh 0}{2} - 0\right)]$

$= \frac{1}{2}\left[\left(\frac{\sinh 2}{2} - 1\right) - \left(\frac{0}{2} - 0\right)\right] = \frac{\sinh 2}{4} - \frac{1}{2} = \frac{\sinh 2 - 2}{4}$.

$3: \int_0^{\ln 2} 4\, e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x \left[\frac{e^x - e^{-x}}{2}\right] dx = 2 \int_0^{\ln 2} (e^{2x} - 1) dx$

$= \int_0^{\ln 2} 2e^{2x}\, dx - \int_0^{\ln 2} 2dx = [e^{2x}]_0^{\ln 2} - [2x]_0^{\ln 2}$

$= \left(e^{2\ln 2} - e^0\right) - (2\ln 2 - 0) = \left(e^{\ln 4} - 1\right) - 2\ln 2$

$= 4 - 1 - 2\ln 2 = 3 - 2\ln 2$.

5: Integrals Leading to Inverse Hyperbolic Functions:

No.	The Integrals
1	$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} + C \,, a > 0$
2	$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\frac{u}{a} + C \ \,, u > a > 0$
3	$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{u}{a} + C \,;\ u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\frac{u}{a} + C;\ u^2 > a^2 \end{cases}$
4	$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\,\mathrm{sech}^{-1}\frac{u}{a} + C \,; 0 < u < a$
5	$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\,\mathrm{csch}^{-1}\frac{u}{a} + C \,; u = 0 \ and \ a > 0$

Ex. 5:

Evaluate the following integrals:

1: $\int \frac{2\,dx}{\sqrt{3+4x^2}}$ **2**: $\int_0^{\frac{1}{2}} \frac{dx}{1-x^2}$

Sol.:

1: $\int \frac{2\,dx}{\sqrt{3+4x^2}}$

$Let\ u = 2x \ \to du = 2\,dx\,; a = \sqrt{3}\,.$

$So, \int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\frac{u}{a} + C = \sinh^{-1}\frac{2x}{\sqrt{3}} + C\,.$

2: $\int_0^{\frac{1}{2}} \frac{dx}{1-x^2} = [\tanh^{-1} x]_0^{\frac{1}{2}} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2} - 0 = \tanh^{-1}\frac{1}{2}\,.$

H.W. 2

I: Evaluate the following integrals:

1: $\int 4\,cosh(3x - ln\,2)dx$ **2**: $\int coth\left(\frac{x}{\sqrt{3}}\right)dx$

3: $\int sech^2\left(x-\frac{1}{2}\right)dx$ 4: $\int \frac{sech\sqrt{x}\,tanh\sqrt{x}}{\sqrt{x}}\,dx$

5: $\int_{-\ln 4}^{-\ln 2} 2e^x \cosh x\,dx$ 6: $\int_0^{\frac{\pi}{2}} 2\sinh(\sin x)\cos x\,dx$

7: $\int_1^2 \frac{\cos(\ln x)}{x}\,dx$ 8: $\int_{-\ln 2}^{0} \cosh^2\frac{x}{2}\,dx$ 9: $\int_0^{\frac{1}{3}} \frac{6\,dx}{\sqrt{1+9x^2}}$

10: $\int_{\frac{5}{4}}^{2} \frac{dx}{1-x^2}$ 11: $\int_1^2 \frac{dx}{x\sqrt{4+x^2}}$ 12: $\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1+\sin^2 x}}$

Lecture 11

Techniques of Integrations 1

1: Basic Integration Formulas:

No.	Formulas
1	$\int 0\, dx = C$
2	$\int k\, dx = k\, x + C$, k is any number
3	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \; ; n \neq -1$
4	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$
5	$\int e^x\, dx = e^x + C$
6	$\int a^x\, dx = \frac{a^x}{\ln a} + C \; ; a > 0, a \neq 1$
7	$\int \sin x\, dx = -\cos x + C$
8	$\int \cos x\, dx = \sin x + C$
9	$\int \sec^2 x\, dx = \tan x + C$
10	$\int \csc^2 x\, dx = -\cot x + C$
11	$\int \sec x \tan x\, dx = \sec x + C$
12	$\int \csc x \cot x\, dx = -\csc x + C$
13	$\int \tan x\, dx = -\ln\lvert\cos x\rvert + C = \ln\lvert\sec x\rvert + C$
14	$\int \cot x\, dx = \ln\lvert\sin x\rvert + C$
15	$\int \sec x\, dx = \ln\lvert\sec x + \tan x\rvert + C$

16	$\int csc\, x\, dx = -ln\lvert csc\, x + cot\, x\rvert + C$
17	$\int sinh\, x\, dx = cosh\, x + C$
18	$\int cosh\, x\, dx = sinh\, x + C$
19	$\int sech^2 x\, dx = tanh\, x + C$
20	$\int csch^2 x\, dx = -coth\, x + C$
21	$\int sech\, x\, tsnh = -sech\, x + C$
22	$\int csch\, x\, coth\, x\, dx = -csch\, x + C$
23	$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
24	$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
25	$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\lvert\frac{x}{a}\rvert + C$
26	$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} + C\,; a > 0$
27	$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C\,; x > a > 0$

2: Integration by Parts:

Integration by parts is a technique for simplifying integrals of the form $\int f(x)g(x)dx$. It is useful when f can be differentiable repeatedly and g can be integrated repeatedly without difficulty.

If f and g are differentiable $functions\ for\ x$, then the derivative of $f(x)g(x)\ is\ given\ by$: $\frac{d}{dx}[f(x)g(x)] = f'(x)\, g(x) + f(x)\, g'(x)$.

Therefore,

$$\int \frac{d}{dx}[f(x)g(x)]dx = \int [f'(x)\, g(x) + f(x)\, g'(x)]\, dx$$

$$= \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

Or

$$\int f(x)\, g'(x)\, dx = \int \frac{d}{dx}[f(x)g(x)]\, dx - \int f'(x)\, g(x)\, dx$$

Hence,

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx\ .$$

The above formula is called the formula of integration by parts.

Sometimes it is easier to remember the formula if we write it in differential form.

$$Let\ u = f(x)\ and\ v = g(x)\ \rightarrow du = f'(x)dx\ and\ dv = g'(x)\, dx\ .$$

Then the above formula becomes:

$$\int udv = uv - \int vdu$$

This formula express one integral, $\int udv$ **, in terms of a second integral,** $\int vdu$ **, with a proper** $choice\ of\ u\ and\ v$**, the second integral may be easier to evaluate than the first integral.**

Ex. 1:

Find $\int x \cos x\, dx$.

Sol.:

$$Let\ u = x\ and\ dv = \cos x\, dx\ \rightarrow du = dx\ and\ v = \sin x\ .$$

$$So, \int x \cos x\, dx = \int u\, dv = uv - \int v\, du$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + C\ .$$

Ex. 2:

Find $\int \ln x\, dx$.

Sol.:

$Let\ u = ln\ x\ \ and\ \ dv = dx\ \rightarrow\ \ du = \frac{1}{x}\ dx\ \ and\ v = x\ .$

So, $\int ln\ x\ dx\ = \int u\ dv = uv - \int v\ du$

$$= x\ ln\ x - \int x\left(\frac{1}{x}\ dx\right) = x\ ln\ x - \int dx = x\ ln\ x - x + C\ .$$

<u>Note 1:</u>

Sometimes we have to use integration by parts more than once, as in the following example.

<u>Ex. 3:</u>

Find $\int x^2 e^x\ dx$.

<u>Sol.:</u>

$Let\ u = x^2\ \ and\ dv = e^x\ dx\ \ \rightarrow du = 2x\ dx\ \ and\ v = e^x$.

$So, \int x^2 e^x\ dx\ = \int u\ dv = uv - \int v\ du$

$$= x^2 e^x - \int 2x\ e^x\ dx = x^2 e^x - 2\int x\ e^x\ dx\ .$$

Now, we integrate by parts again with

$u = x\ \ and\ \ dv = e^x\ dx\ \rightarrow du = dx\ \ and\ \ v = e^x$.

Then,

$\int x^2 e^x\ dx = x^2 e^x - 2[x\ e^x - \int e^x\ dx] = x^2 e^x - 2xe^x + 2e^x + C$.

<u>Ex. 4:</u>

Find $\int e^x\ cos\ x\ dx$.

<u>Sol.:</u>

$Let\ u = e^x\ \ and\ \ dv = cos\ x\ dx\ \rightarrow\ \ du = e^x\ dx\ \ \ and\ \ v = sin\ x$.

$So, \int e^x\ cos\ x\ dx = \int u\ dv = uv - \int v\ du$

$$= e^x\ sin\ x - \int e^x\ sin\ x\ dx$$

Again:

$Let\ u = e^x\ \ and\ \ dv = \sin x\, dx\ \ \rightarrow du = e^x\, dx\ \ \ and\ \ v = -\cos x\ .$

Then,

$\int e^x \cos x\, dx = e^x \sin x - [-e^x \cos x - \int -e^x \cos x\, dx]$

$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx\ .$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give:

$2 \int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1\ .$

Dividing by 2 and renaming the constant of integration give:

$\int e^x \cos x\, dx = \frac{1}{2}\,(e^x \sin x + e^x \cos x) + C\ .$

<u>Another Way:</u>

$Let\ u = \cos x\ \ and\ \ \rightarrow du = -\sin x\, dx\ \ and\ \ v = e^x\ .$

$So, \int e^x \cos x\, dx = \int u\, dv = uv - \int v\, du$

$= e^x \cos x - \int - e^x \sin x\, dx = e^x \cos x + \int e^x \sin x\, dx$

$Now,\ let\ u = \sin x\ \ and\ \ dv = e^x\, dx\ \ \rightarrow du = \cos x\, dx\ \ and\ \ v = e^x\ .$

Then $\int e^x \cos x\, dx = e^x \cos x + [\, e^x \sin x - \int e^x \cos x\, dx]$

$= e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$

Thus $2 \int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1$

Therefore $\int e^x \cos x\, dx = \frac{1}{2}\,(e^x \sin x + e^x \cos x) + C\ .$

<u>Note 2:</u>

The formula of integration by parts for definite integrals is given as follows:

$$\int_a^b f(x)\, g'(x)\, dx = [f(x)\, g(x)]_a^b - \int_a^b f'(x)\, g(x)\, dx$$

<u>Ex. 5:</u>

$Evaluate\ \int_0^4 x\, e^{-x}\, dx\ .$

Sol.:

$Let\ u = x\ \ and\ \ dx = \ e^{-x}\,dx\ \ \rightarrow du = dx\ \ and\ \ v = -e^{-x}\ .$

Then $\int_0^4 x\, e^{-x}\, dx = [\,-x\, e^{-x}]_0^4 - \int_0^4 -e^{-x}\, dx$

$$= [\,-x\, e^{-x}]_0^4 - [e^{-x}]_0^4$$

$$= -4e^{-4} - (e^{-4} - e^{0}) = -4e^{-4} - e^{-4} + 1 = -5e^{-4} + 1\ .$$

3: Tabular Integration:

Ex. 6:

$Find\ \int x^2\, e^x\, dx\ .$

Sol.:

$Let\ \ f(x) = x^2\ \ and\ g(x) = e^x\ .$

Now create the following table:

$f(x)$ and its derivatives	Sign	$g(x)$ and its integrals
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0	$-$	e^x

Then

$\int x^2\, e^x\, dx = x^2\, e^x - 2xe^x + 2e^x + C\ .$

Ex. 7:

$Find\ \int x^3 \sin x\, dx.$

Sol.:

$Let\ f(x) = x^3\ \ and\ g(x) = \sin x\ .$

Now create the following table:

$f(x)$ and its derivatives	Sign	$g(x)$ and its integrals
x^3	$+$	$\sin x$
$3x^2$	$-$	$-\cos x$
$6x$	$+$	$-\sin x$
6	$-$	$\cos x$
0	$+$	$\sin x$

Then

$$\int x^3 \sin x\, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C\ .$$

<u>H.W.1</u>

Evaluate the following integrals using integration by parts:

1: $\int x \sin \frac{x}{2}\, dx$ 2: $\int x^2 \cos x\, dx$ 3: $\int_1^2 x \ln x\, dx$

4: $\int_1^e x^3 \ln x\, dx$ 5: $\int x\, e^{3x}\, dx$ 6: $\int \tan^{-1} x\, dx$

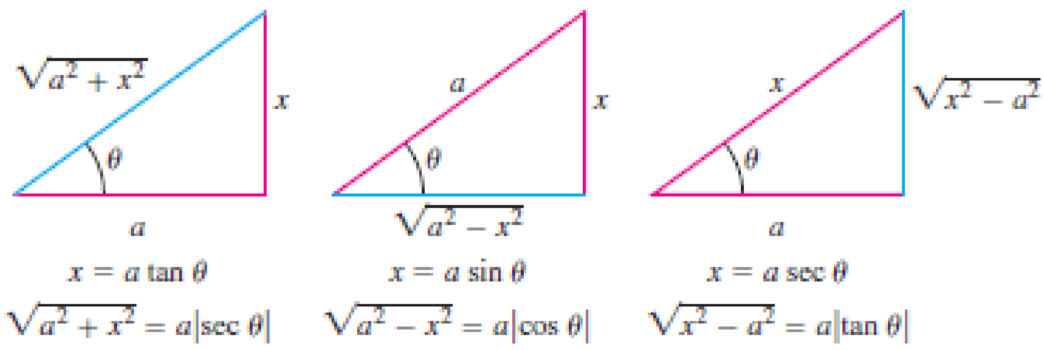
7: $\int e^x \sin x\, dx$ 8: $\int e^{2x} \cos 3x\, dx$ 9: $\int e^{-2x} \sin 2x\, dx$

10: $\int_0^{\frac{\pi}{3}} x \tan^2 x\, dx$ 11: $\int \sin(\ln x)\, dx$ 12: $\int_0^{\frac{1}{\sqrt{2}}} 2x\ \sin^{-1} x^2\, dx$

<u>4: Trigonometric Substitutions:</u>

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function as given in the following table:

Integrand	Substitution	Result after substitution	dx
$a^2 + x^2$	$x = a\tan\theta$	$a^2 \sec^2\theta$	$a\sec^2\theta\, d\theta$
$a^2 - x^2$	$x = a\sin\theta$	$a^2 \cos^2\theta$	$a\cos\theta\, d\theta$
$x^2 - a^2$	$x = a\sec\theta$	$a^2 \tan^2\theta$	$a\sec\theta\tan\theta\, d\theta$



Procedure for a Trigonometric Substitutions

1: Write down the $\boldsymbol{substitution\ for\ x}$**, calculate the differentiable** $\boldsymbol{dx}$**, and specify the selected values of** $\boldsymbol{\theta}$ **for the substitution.**

2: Substitute the trigonometric expression and calculate differential into the integrand, and then simplify the results algebraically.

3: Integrate the trigonometric integral, keeping in mind the restrictions on the angle $\boldsymbol{\theta}$ **for reversibility.**

4: Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the $\boldsymbol{original\ variable\ x}$**.**

Ex. 8:

$\boldsymbol{Find\ \int \frac{dx}{\sqrt{4+x^2}}}$.

Sol.:

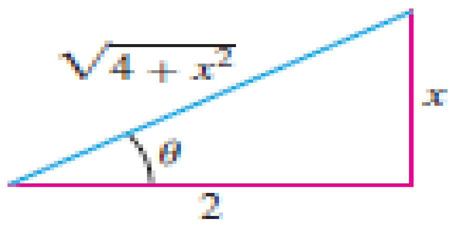
$\boldsymbol{Let\ x = 2\tan\theta \ \rightarrow dx = 2\sec^2\theta\, d\theta\ ;\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$.

$\boldsymbol{So, 4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta}$.

Then $\boldsymbol{\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta}$

$$\boldsymbol{= \ln|\sec\theta + \tan\theta| + C}$$

Note 3: Since $\boldsymbol{-\frac{\pi}{2} < \theta < \frac{\pi}{2}\ , then\ \sec\theta > 0\ and\ |\sec\theta| = \sec\theta}$.



Thus, $\int \frac{dx}{\sqrt{4+x^2}} = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$.

Ex. 9:

$$Find \int \frac{x^2\,dx}{\sqrt{9-x^2}} \quad .$$

Sol.:

$Let\ x = 3\sin\theta \ \rightarrow dx = 3\cos\theta\, d\theta\ ;\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$So,\ 9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta$.

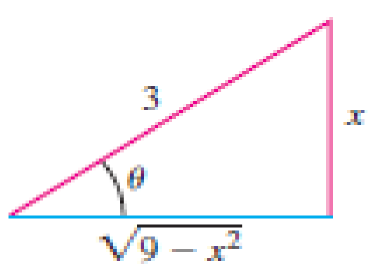
Then $\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{(9\sin^2\theta)(3\cos\theta\,d\theta)}{\sqrt{9\cos^2\theta}} = \int \frac{9\sin^2\theta\cos\theta\,d\theta}{|\cos\theta|}$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \ \rightarrow \cos\theta > 0\ ,\ so\ |\cos\theta| = \cos\theta$.

Therefore, $\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta\cos\theta\,d\theta}{\cos\theta} = 9\int \sin^2\theta\ d\theta = 9\int \frac{1-\cos 2\theta}{2}\ d\theta$

$$= \frac{9}{2}\int d\theta - \frac{9}{2}\int \cos 2\theta\, d\theta = \frac{9}{2}\ \theta - \frac{9}{4}\sin 2\theta + C$$

$$= \frac{9}{2}\ \theta - \frac{9}{4}(2\sin\theta\cos\theta) + C = \frac{9}{2}[\theta - \sin\theta\cos\theta] + C$$



Then $\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \frac{9}{2}[\sin^{-1}\frac{x}{3} - \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C = \frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \frac{x\left(\sqrt{9-x^2}\right)}{9}\right] + C$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{1}{2}x\left(\sqrt{9-x^2}\right) + C\ .$$

Ex. 10:

Find $\int \frac{dx}{\sqrt{25x^2-4}}\ \ ; x > \frac{2}{5}$.

Sol.:

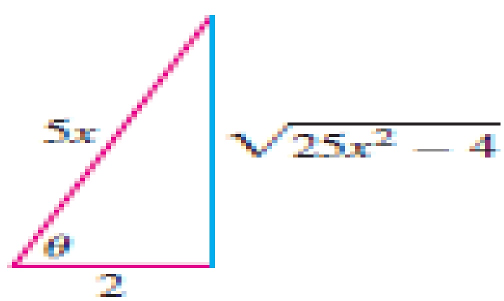
Let $5x = 2\sec\theta \ \rightarrow 5dx = 2\sec\theta\tan\theta\,d\theta \ \rightarrow dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$.

$So, 25x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta\ \ , 0 < \theta < \frac{\pi}{2}$.

Then $\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{\sqrt{4\tan^2\theta}} = \int \frac{\frac{2}{5}\sec\theta\tan\theta\,d\theta}{2|\tan\theta|}$

Since $0 < \theta < \frac{\pi}{2} \ \rightarrow \tan\theta > 0$, so $|\tan\theta| = \tan\theta$.

Therefore $\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\int \sec\theta\,d\theta = \frac{1}{5}\ ln|\sec\theta + \tan\theta| + C$



Then $\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\ ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$.

H.W.2

Evaluate the following integrals:

1: $\int \frac{dx}{\sqrt{9+x^2}}$ 2: $\int_{-2}^{2} \frac{dx}{4+x^2}$ 3: $\int_{0}^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$ 4: $\int_{0}^{\frac{\sqrt{2}}{2}} \frac{2dx}{\sqrt{1-4x^2}}$ 5: $\int \frac{dx}{\sqrt{4x^2-49}}\ , x > \frac{7}{2}$

$6: \int \frac{\sqrt{x^2-25}}{x^3}\, dx\,, x > 5$ $7: \int_0^{\ln 4} \frac{e^t\, dt}{\sqrt{e^{2t}+9}}$ $8: \int_{\frac{1}{12}}^{\frac{1}{4}} \frac{2dt}{\sqrt{t}+4t\sqrt{t}}$ $9: \int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$

$10: \int \frac{\sqrt{1-(\ln x)^2}}{x \ln x}\, dx$.

-

Lecture 12

Techniques of Integrations 2

Integration of Rational Functions by Partial Fractions:

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions called partial fractions which are easily integrated. The method for rewriting rational functions as a sum of simpler fractions is called the method of partial fractions.

Success in writing a rational function $\frac{f(x)}{g(x)}$ as a sum of partial fractions depends on two things:

1: The degree of $f(x)$ must be less than the degree of $g(x)$. If is not, divide $f(x)$ by $g(x)$ and work with remainder term.

2: We must know the factors of $g(x)$.

Method of Partial Fractions $\frac{f(x)}{g(x)}$ **:**

1: Let $x - r$ be a linear $factor\ of\ g(x)$. Suppose that $(x-r)^m$ is the highest $power$ of $x - r\ that\ divides\ g(x)$. Then, to this factor, assign the sum of m partial fractions as:

$$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m} .$$

Do this for each distinct factor of $g(x)$.

2: Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$, so that $x^2 + px + q\ has$ no real roots. Suppose that $(x^2 + px + q\ )^n$ is the highest $power$ of this factor $that\ divides\ g(x)$. Then, to this factor, assign $the\ sum$ of n partial fractions as:

$$\frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}$$

Do this for each distinct quadratic of $g(x)$.

3: Set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions.

Clear the resulting equation of fractions and arrange the terms in $decreasing\ power\ of\ x$.

4: Equate the coefficients of corresponding powers of x and solve the resulting $equations$ for undetermined coefficients.

<u>Ex. 1:</u>

Use partial fractions method $to\ find\ \int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\ dx$.

<u>Sol.:</u>

The partial fractions decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

To find the values of $undetermined\ coefficients\ A, B, and\ C$, we clear fractions and get

$$x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)\ .$$

$$Let\ x = -1\ \rightarrow\ -2 = -4B\ \rightarrow B = \frac{1}{2}\ .$$

$$Let\ x = 1\ \rightarrow 6 = 8A\ \rightarrow A = \frac{6}{8} = \frac{3}{4}\ .$$

$$Let\ x = 0\ \rightarrow 1 = 3A - 3B - C\ \rightarrow 1 = \frac{9}{4} - \frac{3}{2} - C\ \rightarrow C = \frac{9}{4} - \frac{3}{2} - 1 = -\frac{1}{4}\ .$$

Then $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\ dx = \int \frac{A}{x-1}\ dx + \int \frac{B}{x+1}\ dx + \int \frac{C}{x+3}\ dx$

$$= \int \frac{\frac{3}{4}}{x-1}\ dx + \int \frac{\frac{1}{2}}{x+1}\ dx + \int \frac{-\frac{1}{4}}{x+3}\ dx$$

$$= \frac{3}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x+3}$$

$$= \frac{3}{4} ln|x-1| + \frac{1}{2} ln|x+1| - \frac{1}{4} ln|x+3| + K\ ,$$

Where K is the arbitrary $constant\ of\ integration$.

Ex. 2:

$Use\ partial\ fractions\ to\ evaluate\ \int \frac{6x+7}{(x+2)^2}\ dx$

Sol.:

First we express the integrand as a sum of partial fractions with undetermined coefficients as:

$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}\ \ \rightarrow 6x + 7 = A(x + 2) + B$.

$Let\ x = -2\ \rightarrow\ -5 = B$.

$Let\ \ x = 0\ \ \rightarrow\ \ 7 = 2A + B\ \rightarrow 7 = 2A - 5\ \rightarrow 2A = 12\ \rightarrow A = 6$.

Then $\int \frac{6x+7}{(x+2)^2}\ dx = \int \frac{A}{x+2}\ dx + \int \frac{B}{(x+2)^2}\ dx = \int \frac{6}{x+2}\ dx + \int \frac{-5}{(x+2)^2}\ dx$

$$= 6ln|x + 2| + \frac{5}{x+2} + C .$$

Ex. 3:

$Use\ partial\ fractions\ method\ to\ evaluate\ \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\ dx$.

Sol.:

First we divide the denominator into the numerator to get a polynomial plus proper fraction.

$$\begin{array}{r|l} & 2x \\ \hline x^2 - 2x - 3 & 2x^3 - 4x^2 - x - 3 \\ & \underline{2x^3 - 4x^2 - 6x} \\ & 5x - 3 \end{array}$$

Therefore, $\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3} = \frac{5x-3}{(x+1)(x-3)}$.

Now, we take $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}\ \rightarrow 5x - 3 = A(x - 3) + B(x + 1)$

$Let\ x = -1\ \rightarrow\ -8 = -4A\ \rightarrow A = 2$.

$Let\ x = 3\ \rightarrow\ \ 12 = 4B\ \rightarrow B = 3$.

Then $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$

$$x^2 + 2\ln|x+1| + 3\ln|x-3| + C\,.$$

Ex. 4:

Use partial fraction method to evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

Sol.:

First we express the integrand as a sum of partial fractions with undetermined coefficients as:

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \rightarrow$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$Let\ x = 1 \rightarrow 2 = 2D \rightarrow D = 1$.

$Let\ x = 0 \rightarrow 4 = B - C + D \rightarrow$ $B - C = 3$ ……………………….(1)

$Let\ x = -1 \rightarrow 6 = 4(-A+B) - 4C + 2D \rightarrow 6 = -4A + 4B - 4C + 2 \rightarrow$

$3 = -2A + 2B - 2C + 1 \rightarrow -2A + 2B - 2C = 2 \rightarrow$

$-A + B - C = 1$ ………………………(2)

$Let\ x = 2 \rightarrow 0 = (2A+B) + 5C + 5D \rightarrow 0 = 2A + B + 5C + 5 \rightarrow$

$2A + B + 5C = -5$ ……………………(3)

Now, from (1) and (2), we get $A = 2$.

Put $A = 2$ **in equations (2) and (3), we get:**

$B - C = 3$ ………………………………………(4)

$B + 5C = -9$ ………………………………….(5)

From (4) and (5), we get: $C = -2$.

Put $C = -2$ **in (1), we get:** $B = 1$.

Thus, we have $A = 2,\ B = 1,\ C = -2,\ and\ D = 1$.

Then $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1}\,dx + \int \frac{-2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$

$$= \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - 2\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - (x-1)^{-1} + K \, .$$

Ex. 5:

$Use\ partial\ fraction\ method\ to\ evaluate\ \int \frac{dx}{x\,(x^2+1)^2}$.

Sol.:

First we express the integrand as a sum of partial fractions with undetermined coefficients as:

$$\frac{1}{x\,(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \rightarrow$$

$$1 = A(x^2+1)^2 + x(Bx+C)(x^2+1) + x(Dx+E)$$

$Let\ x = 0 \ \rightarrow 1 = A$.

$Let\ x = 1 \ \rightarrow 1 = 4 + 2(B+C) + D + E \ \rightarrow 2B + 2C + D + E = -3 \ldots\ldots (1)$

$Let\ x = -1 \ \rightarrow 1 = 4 - 2(-B+C) - (-D+E) \ \rightarrow$

$1 = 4 + 2B - 2C + D - E \ \rightarrow 2B - 2C + D - E = -3 \ldots\ldots (2)$

$Let\ x = 2 \ \rightarrow 1 = 25 + 10(2B+C) + 2(2D+E) \ \rightarrow$

$1 = 25 + 20B + 10C + 4D + 2E \ \rightarrow 20B + 10C + 4D + 2E = -24 \ \rightarrow$

$10B + 5C + 2D + E = -12 \ldots\ldots (3)$

$Let\ x = -2 \ \rightarrow 1 = 25 - 10(-2B+C) - 2(-2D+E) \ \rightarrow$

$1 = 25 + 20B - 10C + 4D - 2E \ \rightarrow 20B - 10C + 4D - 2E = -24 \ \rightarrow$

$10B - 5C + 2D - E = -12 \ldots\ldots (4)$

$Adding\ (1) and\ (2), we\ get: 4B + 2D = -6 \ \rightarrow 2B + D = -3 \ldots\ldots\ldots\ldots\ldots (5)$

$Adding\ (3) and\ (4), we\ get: 20B + 4D = -24 \rightarrow 5B + D = -6 \ldots\ldots\ldots\ldots (6)$

From (5) and (6), we get: $B = -1$.

From (5), we get $D = -1$.

Put $B = -1$ **and** $D = -1$ **in (1), we get** $2C + E = 0 \ \rightarrow E = -2C \ldots\ldots\ldots\ldots\ (7)$

Put $B=-1$ and $D=-1$ and $E=-2C$ in (3), we get $C=0$ and $E=0$.

Therefore, we have $A=1, B=-1, C=0, D=-1, and\ E=0$.

Then $\int \frac{dx}{x\,(x^2+1)^2} = \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$

$$= ln|x| - \frac{1}{2}ln(x^2+1) + \frac{1}{2(x^2+1)} + K\ .$$

H.W. 1

1: $\int \frac{dx}{1-x^2}$ 2: $\int \frac{dx}{x^2+2x}$ 3: $\int \frac{2x+1}{x^2-7x+12}dx$ 4: $\int \frac{x+3}{2x^3-8x}\,dx$

5: $\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$ 6: $\int \frac{dx}{(x^2-1)^2}$ 7: $\int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$ 8: $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$

9: $\int \frac{x^2\,dx}{x^4-1}$ 10: $\int \frac{x^2+x}{x^4-3x^2-4}\,dx$ 11: $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$ 12: $\int \frac{sin\,t\,dt}{cos^2 t+cos\,t-2}$

Improper Integrals:

1: Improper Integral of Type I:

Integrals with infinite limits of integration are improper integral of Type I.

1: $If\ f(x)\ is\ continuous\ on\ [a,\infty], then\ \int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$.

2: $If\ f(x)\ is\ continuous\ on\ [-\infty,b], then\ \int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$.

3: $If\ f(x)\ is\ continuous\ on\ [-\infty,\infty], then$

$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx\ ; where\ c\ is\ any\ real\ number$.

In each case, if the limit is finite we say that the improper integral converges and that the limit is the value of the improper integral.

If the limit fails to exist, the improper integral diverges.

Ex. 6:

$Evaluate\ \int_1^\infty \frac{ln\,x}{x}\,dx$.

Sol.:

$$\int_1^\infty \frac{\ln x}{x}\, dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\, dx = \lim_{b\to\infty}[\frac{1}{2}\,(\ln x)^2]_1^b = \frac{1}{2}\lim_{b\to\infty}[(\ln b)^2 - (\ln 1)^2]$$

$$= \frac{1}{2}\lim_{b\to\infty}[\,(\ln b)^2 - 0] = \frac{1}{2}\lim_{b\to\infty}(\ln b)^2 = \infty\,.$$

Therefore, this integral diverges.

Ex. 7:

$$Evaluate\ \int_1^\infty \frac{\ln x}{x^2}\, dx\ .$$

Sol.:

$$\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\, dx\,.$$

$$Integrate\ \int_1^b \frac{\ln x}{x^2}\, dx\ \ by\ parts\ as\ follows:$$

$$Let\ u = \ln x\ and\ dv = \frac{1}{x^2}dx\ \to du = \frac{1}{x}\, dx\ \ and\ \ v = -\frac{1}{x}\,.$$

Therefore, $\int_1^b \frac{\ln x}{x^2}\, dx = \int_1^b u\, dv = [uv]_1^b - \int_1^b v\, du$

$$= [-\frac{\ln x}{x}]_1^b - \int_1^b -\frac{1}{x^2}\, dx = \left[-\frac{\ln x}{x}\right]_1^b - [\frac{1}{x}]_1^b$$

$$= -\left[\frac{\ln b}{b} - \frac{\ln 1}{1}\right] - \left[\frac{1}{b} - 1\right] = -\frac{\ln b}{b} - \frac{1}{b} + 1\,.$$

Then $\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty}[-\frac{\ln b}{b} - \frac{1}{b} + 1]$

$$= -\lim_{b\to\infty}\frac{\ln b}{b} - \lim_{b\to\infty}\frac{1}{b} + \lim_{b\to\infty} 1$$

$$= -\lim_{b\to\infty}\frac{\frac{1}{b}}{1} - \lim_{b\to\infty}\frac{1}{b} + \lim_{b\to\infty} 1 = -0 - 0 + 1 = 1\,.$$

Ex. 8:

$$Evaluate\ \int_{-\infty}^\infty \frac{dx}{1+x^2}\ .$$

Sol.:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{c} \frac{dx}{1+x^2} + \int_{c}^{\infty} \frac{dx}{1+x^2} \text{ , where } c \text{ is any real number.}$$

We can choose $c = 0$.

Then

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \quad .$$

Take the first integral:

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a \to -\infty} [\tan^{-1} x]_a^0 = \lim_{a \to -\infty} [\tan^{-1} 0 - \tan^{-1} a]$$

$$= \lim_{a \to -\infty} [0 - \tan^{-1} a] = - \lim_{a \to -\infty} \tan^{-1} a = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

Now take the second integral:

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b \to \infty} [\tan^{-1} x]_0^b = \lim_{b \to \infty} [\tan^{-1} b - \tan^{-1} 0]$$

$$= \lim_{b \to \infty} [\tan^{-1} b - 0] = \lim_{b \to \infty} \tan^{-1} b = \frac{\pi}{2}.$$

Thus $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

2: Improper Integral of Type II:

Integrals of functions that becomes infinite at a point within the interval of integration are improper integrals of Type II.

1: $If\ f(x)\ is\ continuous\ on\ (a,b]\ and\ discontinuous\ at\ a, then$:

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx .$$

2: $If\ f(x)\ is\ continuous\ on\ [a,b)\ and\ discontinuous\ at\ b, then$:

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx .$$

3: $If\ f(x)\ is\ discontinuous\ at\ c, where\ a < c < b\ , and\ continuous\ on$

$[a,c) \cup (c,b], then$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

In each case, if the limit is finite we say that the improper integral converges and that limit is the value of the improper integral.

If the limit does not exist, the integral diverges.

Note 1:

In the part 3 above, the integral on the left side of the equation converges if both integrals on the right side converges; otherwise it diverges.

Ex. 9:

Investigate the convergence of $\int_0^1 \frac{dx}{1-x}$.

Sol.:

The integrand $f(x) = \frac{1}{1-x}$ **is continuous on** $[0,1)$ **but it is discontinuous at** $x = 1$ **and becomes infinite** $as\ x \to 1^-$.

We can evaluate integral as

$$\int_0^1 \frac{dx}{1-x} = \lim_{b \to 1^-} \int_0^b \frac{dx}{1-x} = \lim_{b \to 1^-} [-ln|1-x|]_0^b = -\lim_{b \to 1^-} [ln|1-b| - ln|1-0|]$$

$$= -\lim_{b \to 1^-} |1-b| = -\infty\,.$$

The limit is infinite, so the integral diverges.

Ex. 10:

Evaluate $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

Sol.:

The integrand $f(x) = \frac{1}{(x-1)^{\frac{2}{3}}}$ **is continuous on** $[0,1) \cup (1,3]$.

$$Therefore,\ \int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}\,.$$

Now, evaluate each improper integral on the right- hand side of above equation.

$$\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-}\int_0^b (x-1)^{-\frac{2}{3}}\,dx = \lim_{b\to 1^-}[3(x-1)^{\frac{1}{3}}]_0^b$$

$$= 3\lim_{b\to 1^-}\left[(b-1)^{\frac{1}{3}} - (-1)^{\frac{1}{3}}\right] = 3\lim_{b\to 1^-}\left[(b-1)^{\frac{1}{3}} + 1\right] = 3\,.$$

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{a\to 1^+}\int_a^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{a\to 1^+}\int_a^3 (x-1)^{-\frac{2}{3}}\,dx = \lim_{a\to 1^+}[3(x-1)^{\frac{1}{3}}]_a^3$$

$$= 3\lim_{a\to 1^+}\left[(2)^{\frac{1}{3}} - (a-1)^{\frac{1}{3}}\right] = 3(2)^{\frac{1}{3}} = 3\sqrt[3]{2}\,.$$

$$Thus,\ \int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3\sqrt[3]{2}\,.$$

<u>H.W. 2</u>

Evaluate the following integrals without using tables:

1: $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$ 2: $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ 3: $\displaystyle\int_{-1}^1 \frac{dx}{x^{\frac{2}{3}}}$ 4: $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

5: $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$ 6: $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$ 7: $\displaystyle\int_1^\infty \frac{e^x dx}{\sqrt{e^x-1}}$

8: $\int_2^\infty \frac{dx}{x\ln x}$ 9: $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$ 10: $\int_{-\infty}^\infty \frac{x^3 dx}{\sqrt{x^4+1}}$

Lecture 13

Applications of Definite Integrals

1: Volume by Disk Method:

Definition 1:

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a sold of revolution.

Definition 2:

A cross-section of a solid S is the plane region formed by intersecting S with a plane.

To find the volume of a solid we need only observe that the cross-sectional area is the area of a disk $of\ radius\ R(x)$, the distance of the planar region's boundary from the axis of revolution. The area i$s\ then\ A(x) = \pi[r(x)]^2$.

So, the definition of volume in this case gives:

$The\ volume\ by\ Disk\ for\ rotation\ About\ x - axis$:

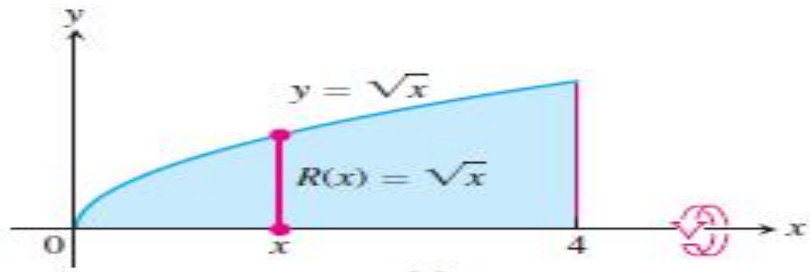
$$V = \int_a^b A(x)dx = \int_a^b \pi[r(x)]^2 dx\ .$$

This method for calculating the volume of a solid of revolution is often called the disk method because a cross-section is a circular disk $of\ radius\ R(x)$.

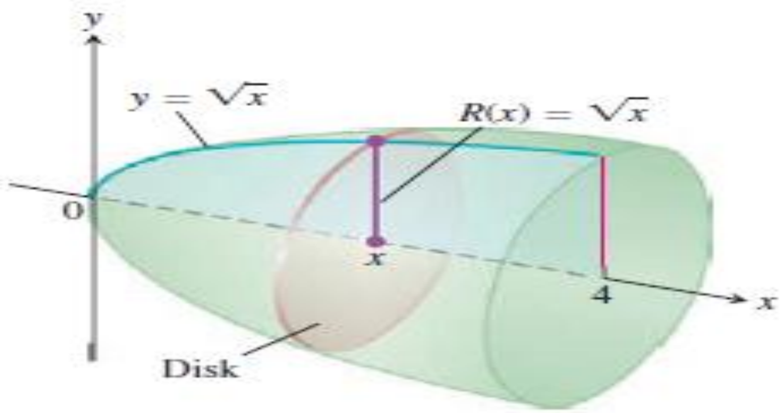
Example 1:

The region between the curve $y = \sqrt{x}\ , 0 \le x \le 4, and\ x - axis$ is revolved about $the\ x - axis$ to generate a solid. Find its volume.

Solution:



(a)



$$V = \int_a^b \pi[r(x)]^2 dx = \pi \int_0^4 [\sqrt{x}]^2 \, dx = \pi \int_0^4 x \, dx = \pi[\tfrac{1}{2}x^2]_0^4 =$$
$$= 8\pi \text{ cubic units.}$$

Example 2:

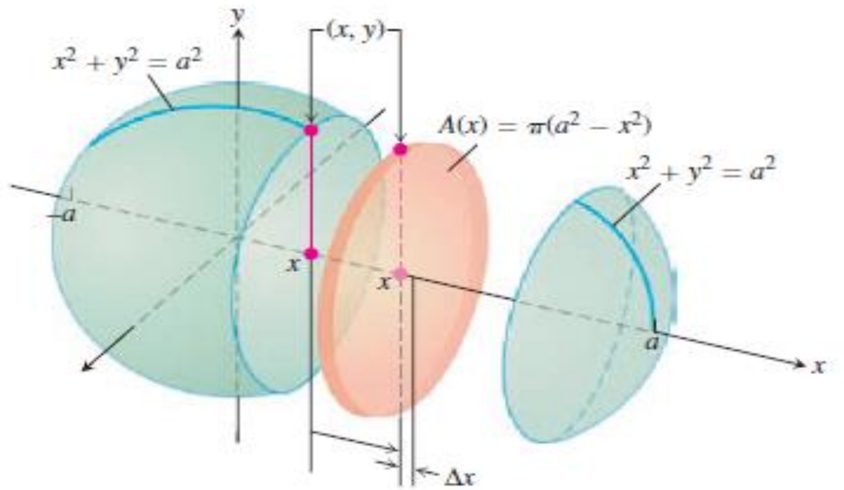
The circle $x^2 + y^2 = a^2$ is rotated about $the\ x - axis$ to generate a sphere. Find its volume.

Solution:

$$V = \int_{-a}^{a} \pi[R(x)]^2 \, dx = \pi \int_{-a}^{a} [\sqrt{a^2 - x^2}\ ]^2 \, dx = \pi \int_{-a}^{a} (a^2 - x^2) dx$$

$$= \pi[a^2 x - \tfrac{1}{3}x^3]_{-a}^{a} = \pi[\left(a^3 - \tfrac{1}{3}a^3\right) - \left(-a^3 + \tfrac{1}{3}a^3\right)]$$

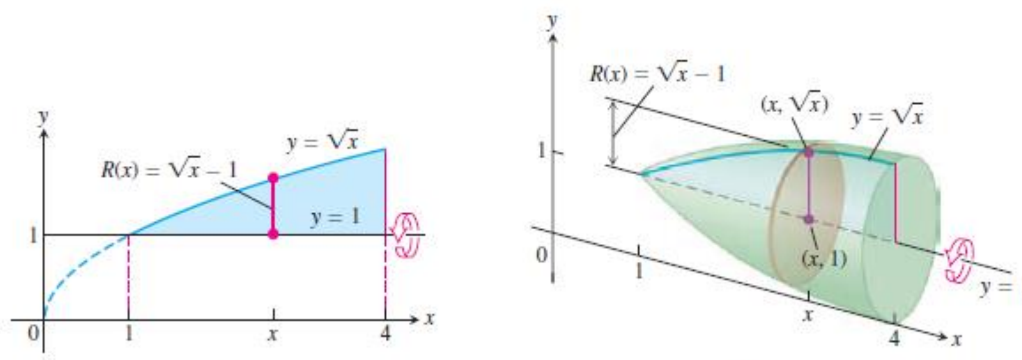
$$= \pi\left[2a^3 - \tfrac{2}{3}a^3\right] = \tfrac{4}{3}\pi a^3 \text{ cubic units.}$$



Example 3:

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ *and the lines* $y = 1, x = 4$ *about the line* $y = 1$.

Solution:



$$V = \int_1^4 \pi[R(x)]^2\,dx = \pi\int_1^4 [\sqrt{x}-1]^2\,dx = \pi\int_1^4 (x - 2\sqrt{x} + 1)dx$$

$$= \pi[\tfrac{1}{2}x^2 - \tfrac{4}{3}x^{\frac{3}{2}} + x)_1^4 = \pi[\left(8 - \tfrac{32}{3} + 4\right) - \left(\tfrac{1}{2} - \tfrac{4}{3} + 1\right)]$$

$$= \pi\left[\left(\tfrac{48}{6} - \tfrac{64}{6} + \tfrac{24}{6}\right) - \left(\tfrac{3}{6} - \tfrac{8}{6} + \tfrac{6}{6}\right)\right] = \pi[\tfrac{8}{6} - \tfrac{1}{6}] = \tfrac{7}{6}\pi \textit{ cubic units}.$$

Note 1:

To find the volume of a solid generated by revolving a region between $y-axis$ and the curve $x = R(y), c \le y \le d$ $about\ the\ y-axis$, w use the sme method with x $replaced\ by\ y$. In this case, the circular cross-section is

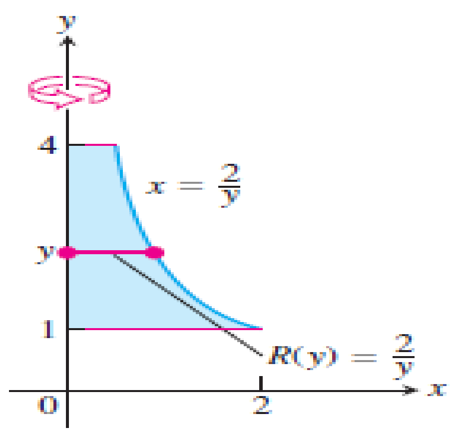
$A(y) = \pi[R(y)]^2$, and **the definition of volume gives by:**

$V = \int_c^d A(y)dy = \int_c^d \pi\,[R(y)]^2\,dy.$

Example 4:

Find the volume of the solid generated by revolving the region between the $y-axis$ and the curve $x = \frac{2}{y}$, $1 \le y \le 4$, about the $y-axis$.

Solution:



$$V = \int_c^d \pi\,[R(y)]^2\,dy = \pi\int_1^4 \frac{4}{y^2}\,dy = \pi\int_1^4 4y^{-2}\,dy = \pi[-4y^{-1}]_1^4$$

$$= \pi[-1+4] = 3\pi\ cubic\ units\ .$$

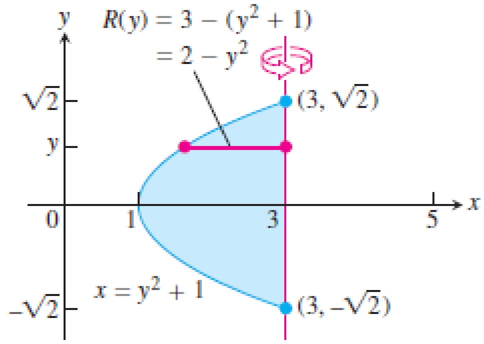
Example 5:

Find the volume of the solid generated by revolving the region between the $parabola\ \ x = y^2 + 1\ and\ the\ line\ x = 3\ about\ the\ line\ x = 3$.

Solution:

First, we find the limits of integration as follows:

$Since\ x = y^2 + 1\ \ and\ x = 3\ , then\ y^2 + 1 = 3 \to y^2 = 2 \to y = \mp\sqrt{2}\ .$



Here, $R(y) = 3 - (y^2 + 1) = 2 - y^2$.

Then $V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\ dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2)^2\ dy$

$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 4y^2 + y^4)dy = \pi[4y - \frac{4}{3}y^3 + \frac{1}{5}y^5)_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \pi[\left(4\sqrt{2} - \frac{8}{3}\sqrt{2} + \frac{4}{5}\sqrt{2}\right) - \left(-4\sqrt{2} + \frac{8}{3}\sqrt{2} - \frac{4}{5}\sqrt{2}\right)]$$

$$= \pi\left[8\sqrt{2} - \frac{16}{3}\sqrt{2} + \frac{8}{5}\sqrt{2}\right] = \pi[\frac{120}{15}\sqrt{2} - \frac{80}{15}\sqrt{2} + \frac{24}{15}\sqrt{2}]$$

$$= \frac{64}{15}\sqrt{2}\,\pi \quad cubic\ units.$$

2: Volume by Washer Method:

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it. The cross-sections perpendicular to the axis of revolution are washers instead of disk. The dimensions of a typical washer are:

$Outer\ radius : R(x)\ \ and\ Inner\ radius: r(x).$

$The\ washer's\ area\ is:\ A(x) = \pi[R(x)^2 - r(x)^2].$

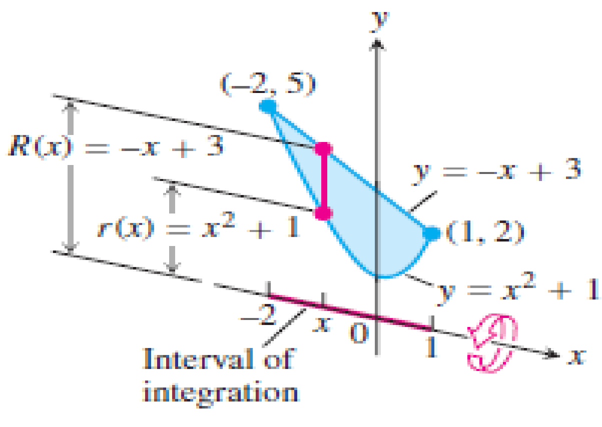
Then the volume is given by:

$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)^2 - r(x)^2]dx\,.$$

<u>Example 6:</u>

The region bounded by the curve $y = x^2 + 1$ **and the line** $y = -x + 3$ **is revolved about** $the\ x - axis$ **to generate a solid. Find the volume of the solid.**

Solution:



First, we find the limits of integration as follows:

Put $x^2 + 1 = -x + 3 \rightarrow x^2 + x - 2 = 0 \rightarrow (x + 2)(x - 1) = 0$

$\rightarrow \; x = -2 \,, x = 1$.

The outer radius is $R(x) = -x + 3$.

The inner radius is $r(x) = x^2 + 1$.

Then $V = \pi \int_{-2}^{1} [(-x + 3)^2 - (x^2 + 1)^2] dx$

$$= \pi \int_{-2}^{1} [(x^2 - 6x + 9) - (x^4 + 2x^2 + 1)] dx$$

$$= \pi \int_{-2}^{1} [-x^4 - 3x^2 - 6x + 8] \, dx$$

$$= \pi \left[-\frac{1}{5} x^5 - x^3 - 3x^2 + 8x \right]_{-2}^{1}$$

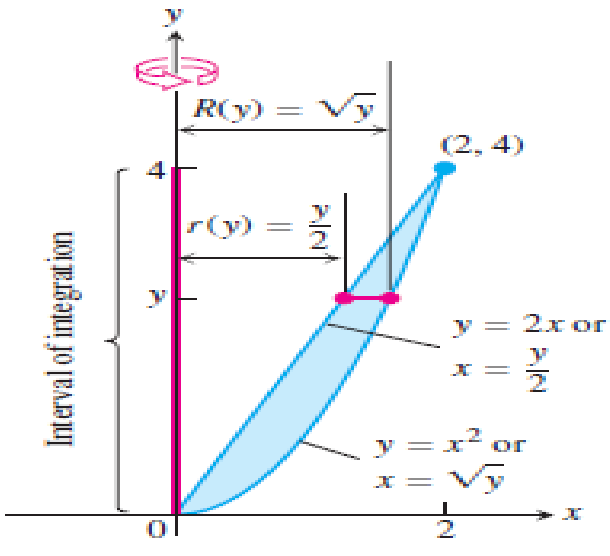
$$= \pi [\left(-\frac{1}{5} - 1 - 3 + 8 \right) - \left(\frac{32}{5} + 8 - 12 - 16 \right)]$$

$$= \pi \left[-\frac{33}{5} + 24 \right] = \frac{117}{5} \pi \;\; cubic\; units \, .$$

Example 7:

The region bounded by the parabola $y = x^2$ **and the line** $y = 2x$ **in the first quadrant is revolved about** $the\ y - axis$ **to generate a solid. Find the volume of this solid.**

Solution:



First, we find the limits of integration as follows:

Put $\sqrt{y} = \frac{y}{2} \to y = \frac{1}{4}y^2 \to 4y = y^2 \to y^2 - 4y = 0 \to y(y-4) = 0$

$\to y = 0, y = 4$.

$\boldsymbol{Outer\ Radius}$: $R(y) = \sqrt{y}$ $\boldsymbol{and\ Innet\ Radius}$: $r(y) = \frac{y}{2}$.

Then $V = \pi \int_0^4 \left[(\sqrt{y})^2 - (\frac{y}{2})^2\right] dy = \pi \int_0^4 \left[y - \frac{1}{4}y^2\right] dy$

$$= \pi[\frac{1}{2}y^2 - \frac{1}{12}y^3]_0^4 = \pi\left[8 - \frac{64}{12}\right] = \pi\left[8 - \frac{16}{3}\right] = \frac{8}{3}\pi\ cubic\ units.$$

H.W. 1

I: Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the $x - axis$.

1: $y = x^2, y = 0, x = 2$ 2: $y = x^3, y = 0, x = 2$

3; $y = \sqrt{9 - x^2}$, $y = 0$ 4: $y = sec\ x$, $y = 0, y = -\frac{\pi}{4}$, $y = \frac{\pi}{4}$.

II: Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the $y - axis$.

1: **The region enclosed by** $x = \sqrt{5}y^2, x = 0, y = -1, y = 1$.

2: The region enclosed by $x = y^{\frac{3}{2}}$, $x = 0, y = 2$.

3: The region enclosed by $x = \sqrt{2\ sin\ 2y}$, $0 \leq y \leq \frac{\pi}{2}$, $x = 0$.

III: Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the $x - axis$.

1: $y = x$, $y = 1, x = 0$ **2**; $y = x^2 + 1, y = x + 3$

3: $y = 4 - x^2$, $y = 2 - x$ **4**: $y = sec\ x, y = tanx, x = 0, x = 1$.

IV: Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the $y - axis$.

1: The region enclosed by the triangle with vertices $(1, 0), (2, 1) and\ (1, 1)$.

2: The region in the first quadrant bounded above by the $parabola\ y = x^2$, below by the $x - axis$, and on the write by the line $x = 2$.

3: Volume Using Cylindrical Shells:

The volume of the solid generated by revolving the region between $the\ x - axis$ and the graph of a continuous function $y = f(x) \geq 0, L \leq a \leq x \leq b$, about the vertical line $x = L$ is given by:

$$V = \int_a^b 2\pi(Shell\ Radius)(Shell\ Height)\ dx .$$

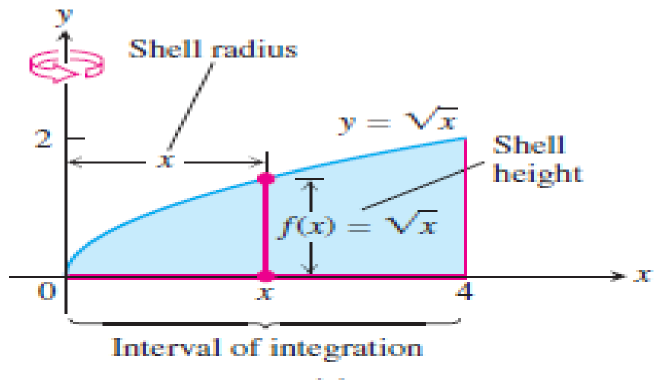
Example 8:

The region bounded by the curve $y = \sqrt{x}$, $the\ x - axis, and\ the\ line\ x = 4$ is revolved about $y - axis$ to generate a solid. Find the volume of a solid.

Solution:

$$V = \int_a^b 2\pi(Shell\ Radius)(Shell\ Height)\ dx$$

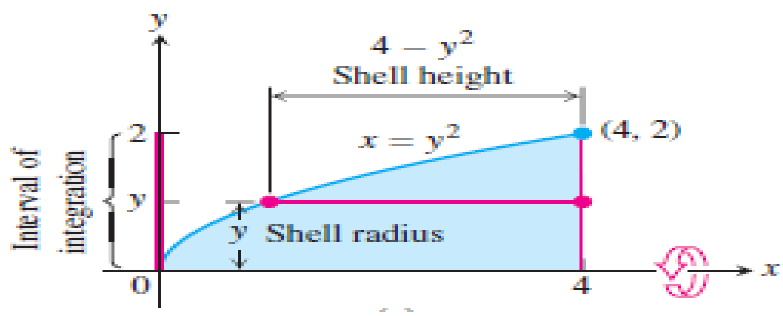
$$= 2\pi \int_0^4 x(\sqrt{x})\ dx = 2\pi \int_0^4 x^{\frac{3}{2}}\ dx = 2\pi[\frac{2}{5}x^{\frac{5}{2}}]_0^4 = \frac{4}{5}\pi[2^5] = \frac{128}{5}\pi\ cubic\ units$$



Example 9:

The region bounded by the curve $y = \sqrt{x}$, $the\ x - axis\ and\ the\ line\ x = 4$ is revolved about $the\ x - axis$ to generate a solid. Find the volume of the solid by shell method.

Solution:



$$V = \int_a^b 2\pi(Shell\ Radius)(Shell\ Height)\ dy$$

$$= 2\pi \int_0^2 y(4 - y^2)dy = 2\pi \int_0^2 (4y - y^3)dy = 2\pi[2y^2 - \frac{1}{4}y^4]_0^2$$

$$= 2\pi[8 - 4] = 8\pi\ \ cubic\ units.$$

Note 2:

If the axis of revolution is the $x - axis$, the limits of integration depend on y.

If the axis of revolution is the $y - axis$, the limits of integration depend on x.

H.W.

I: Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about the $y - axis$.

1: $y = x, y = -\frac{1}{2}x, x = 2$ **2**: $y = x^2, y = 2 - x, x = 0, for\ x \geq 0$.

3: $y = 2 - x^2, y = x^2, x = 0$ **4**: $y = 2x - 1, y = \sqrt{x}, x = 0$.

II: Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about the $x - axis$.

1: $x = \sqrt{y}, x = -y, y = 2$ **2**: $x = y^2, x = -y, y = 2, y \geq 0$.

3: $y = x, y = 2x, y = 2$ **4**: $y = \sqrt{x}\ , y = 0, y = x - 2$.