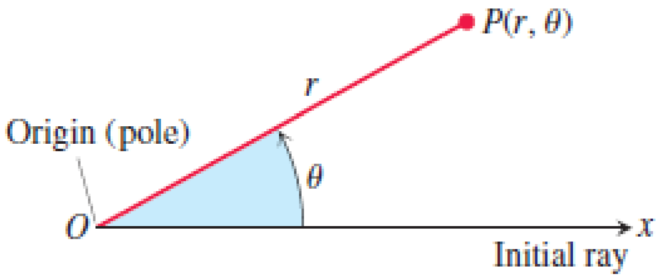
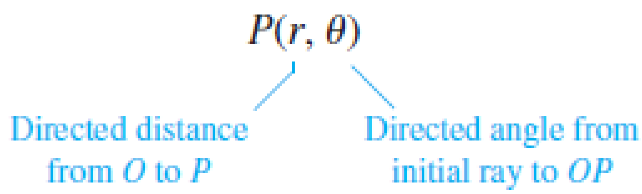
# Chapter One

## Polar Coordinates

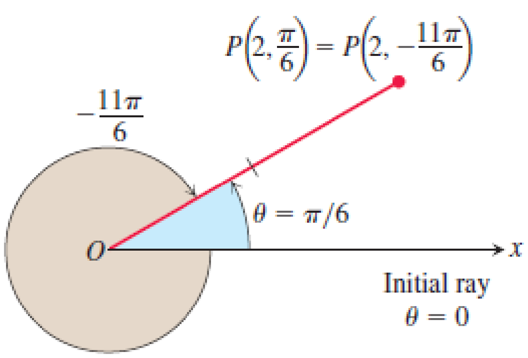


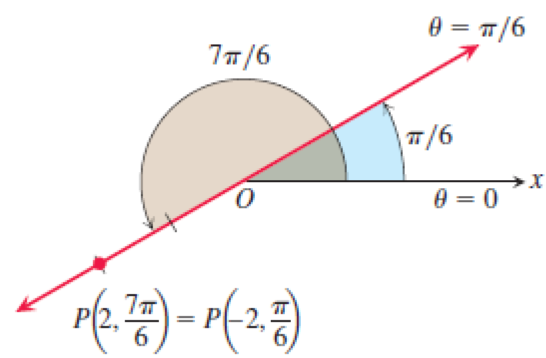

To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.



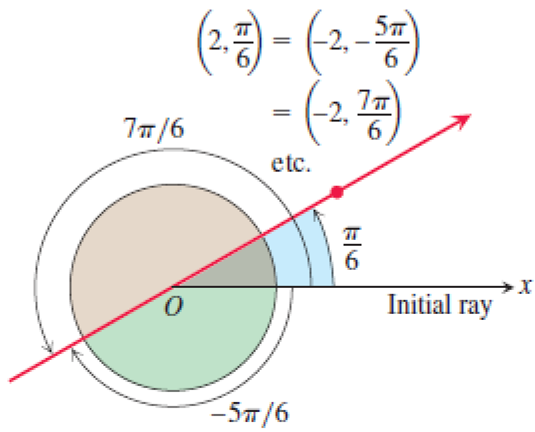

| |
|---|
| **As in trigonometry, $\theta$ is positive when measured counterclockwise and negative when measured clockwise.** |

**EXAMPLE 1:** Find all the polar coordinates of the point $P\ (2, \pi/6)$.



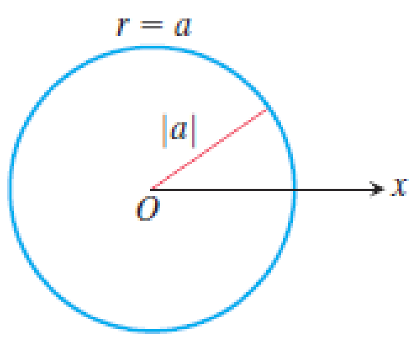




Polar coordinates can have negative $r$-values.

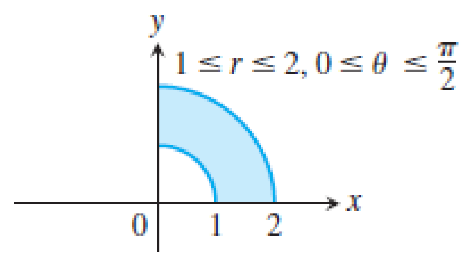


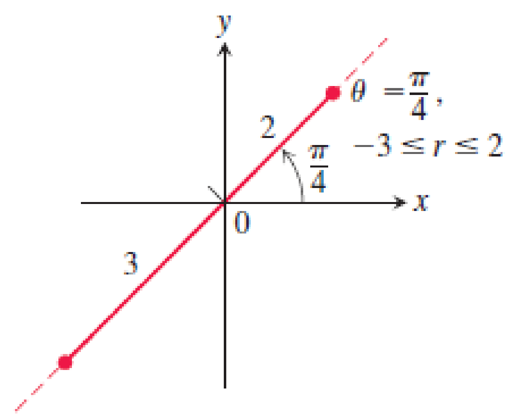

The point $P(2, \pi/6)$
has infinitely many polar coordinate pairs

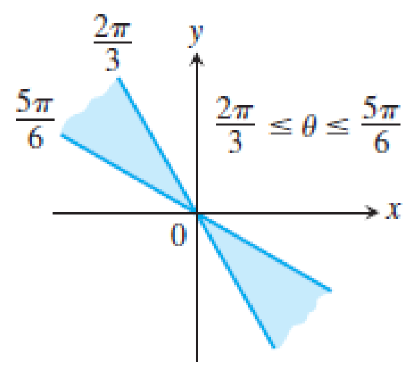
# Polar Equations and Graphs



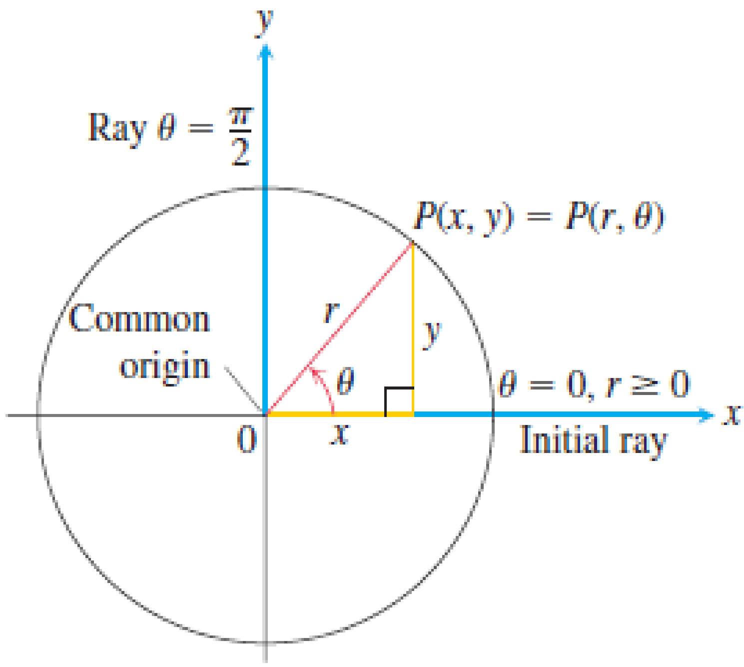

The polar equation for a circle is $r = a$.



# Relating Polar and Cartesian Coordinates




**Equations Relating Polar and Cartesian Coordinates**

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

EXAMPLE 2:

| Polar equation | Cartesian equivalent |
|---|---|
| $r\cos\theta = 2$ | $x = 2$ |
| $r^2\cos\theta\sin\theta = 4$ | $xy = 4$ |
| $r^2\cos^2\theta - r^2\sin^2\theta = 1$ | $x^2 - y^2 = 1$ |
| $r = 1 + 2r\cos\theta$ | $y^2 - 3x^2 - 4x - 1 = 0$ |
| $r = 1 - \cos\theta$ | $x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$ |

EXAMPLE 3: Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

$$x^2 + (y - 3)^2 = 9$$

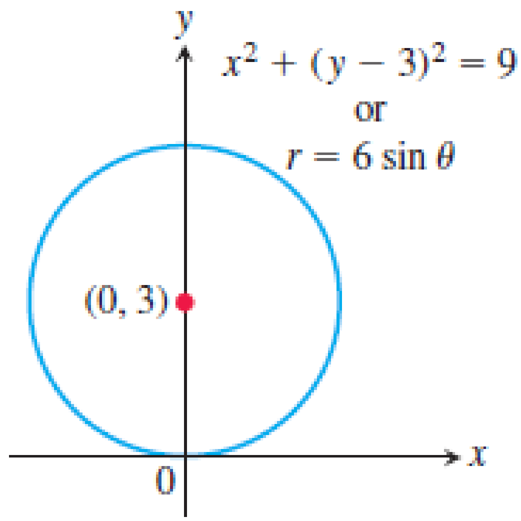
$$x^2 + y^2 - 6y + 9 = 9 \qquad \text{Expand } (y - 3)^2.$$

$$x^2 + y^2 - 6y = 0 \qquad \text{Cancelation}$$

$$r^2 - 6r \sin\theta = 0 \qquad x^2 + y^2 = r^2,\ y = r\sin\theta$$

$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0$$

$$r = 6\sin\theta \qquad \text{Includes both possibilities}$$




EXAMPLE 4: Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

**(a)** $r\cos\theta = -4$

**(b)** $r^2 = 4r\cos\theta$

**(c)** $r = \dfrac{4}{2\cos\theta - \sin\theta}$

**Solution** We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

**(a)** $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$ Substitute.

The graph: Vertical line through $x = -4$ on the $x$-axis

**(b)** $r^2 = 4r\cos\theta$

The Cartesian equation: $r^2 = 4r\cos\theta$

$x^2 + y^2 = 4x$ Substitute.

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$ Complete the square.

$(x - 2)^2 + y^2 = 4$ Factor.

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

**(c)** $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation: $r(2\cos\theta - \sin\theta) = 4$

$2r\cos\theta - r\sin\theta = 4$ Multiply by $r$.

$2x - y = 4$ Substitute.

$y = 2x - 4$ Solve for $y$.

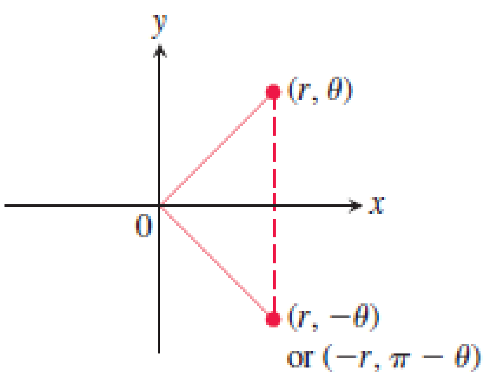
The graph: Line, slope $m = 2$, $y$-intercept $b = -4$

# Graphing Polar Coordinate Equations

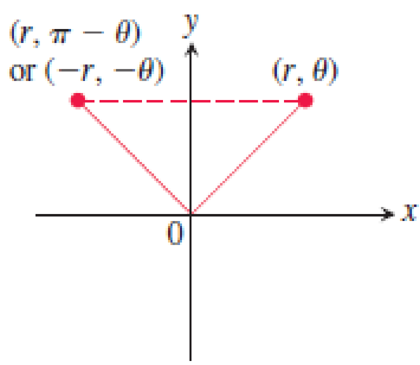
## Symmetry

**Symmetry Tests for Polar Graphs in the Cartesian $xy$-Plane**

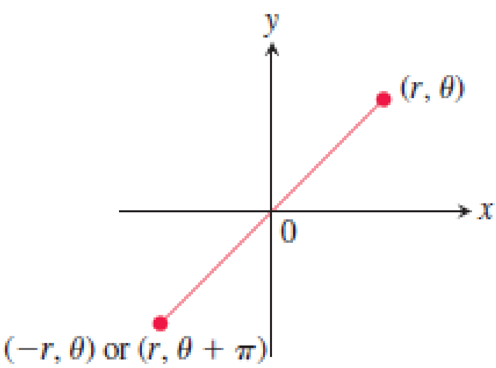
1. *Symmetry about the x-axis:* If the point $(r, \theta)$ lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph
2. *Symmetry about the y-axis:* If the point $(r, \theta)$ lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph
3. *Symmetry about the origin:* If the point $(r, \theta)$ lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph




(a) About the $x$-axis




(b) About the $y$-axis



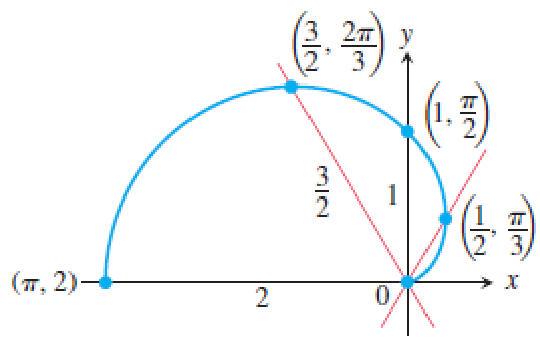

(c) About the origin

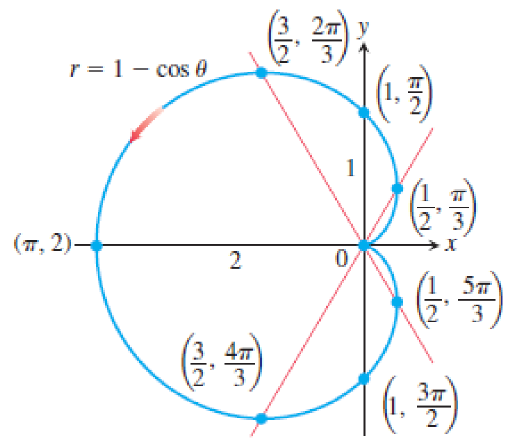
**EXAMPLE 5:** Graph the curve $r = 1 - \cos\theta$ in the Cartesian $xy$-plane.

**Solution** The curve is symmetric about the $x$-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\ &\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

| $\theta$ | $r = 1 - \cos\theta$ |
|---|---|
| $0$ | $0$ |
| $\frac{\pi}{3}$ | $\frac{1}{2}$ |
| $\frac{\pi}{2}$ | $1$ |
| $\frac{2\pi}{3}$ | $\frac{3}{2}$ |
| $\pi$ | $2$ |




**EXAMPLE 6:** Graph the curve $r^2 = 4\cos\theta$ in the Cartesian $xy$-plane.

**Solution** The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running $\theta$ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the $x$-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the $y$-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

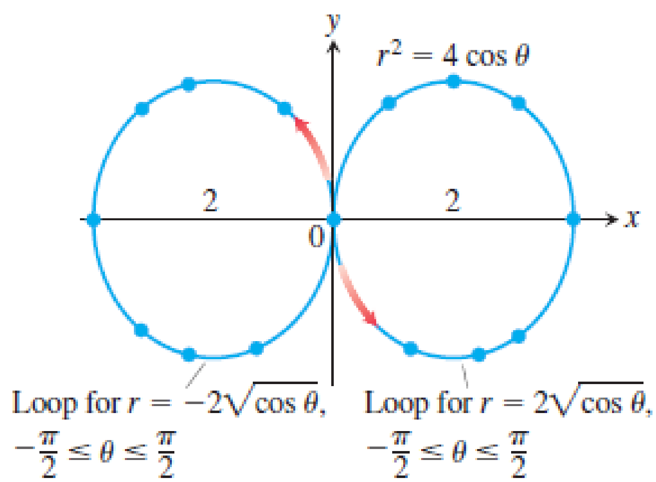
For each value of $\theta$ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of $r$:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 11.30).

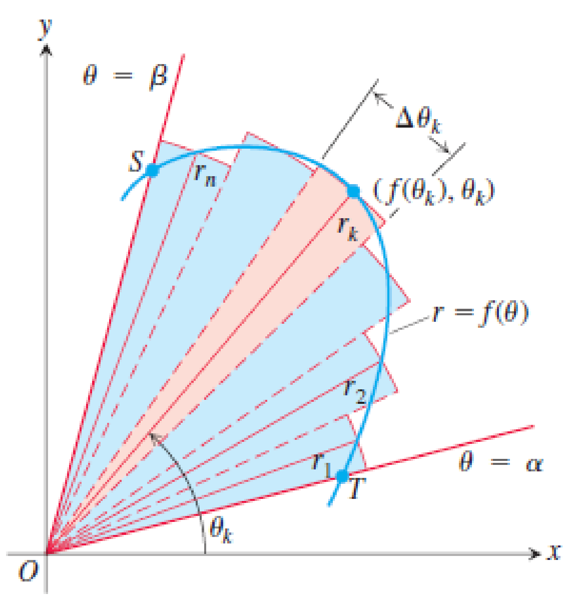
| $\boldsymbol{\theta}$ | $\cos\boldsymbol{\theta}$ | $r = \pm 2\sqrt{\cos\boldsymbol{\theta}}$ |
|---|---|---|
| $0$ | $1$ | $\pm 2$ |
| $\pm\frac{\pi}{6}$ | $\frac{\sqrt{3}}{2}$ | $\approx \pm 1.9$ |
| $\pm\frac{\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\approx \pm 1.7$ |
| $\pm\frac{\pi}{3}$ | $\frac{1}{2}$ | $\approx \pm 1.4$ |
| $\pm\frac{\pi}{2}$ | $0$ | $0$ |

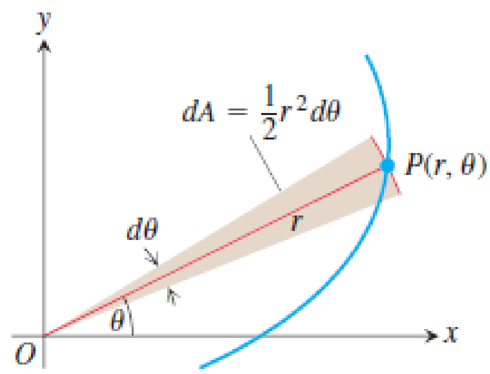
(a)




(b)

# Areas in Polar Coordinates



**Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$ when $\alpha \le \theta \le \beta$, $r \ge 0$, and $\beta - \alpha \le 2\pi$.**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the **area differential**

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left( f(\theta) \right)^2 d\theta.$$

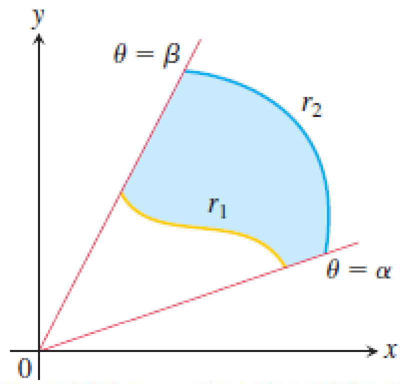
**EXAMPLE 7:** Find the area of the region in the $xy$-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

**Solution** We graph the cardioid and determine that the radius $OP$ sweeps out the region exactly once as $\theta$ runs from 0 to $2\pi$. The area is therefore

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \int_0^{2\pi} \left( 2 + 4\cos\theta + 2 \cdot \frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \\
&= \left[ 3\theta + 4\sin\theta + \frac{\sin 2\theta}{2} \right]_0^{2\pi} = 6\pi - 0 = 6\pi.
\end{aligned}$$

**Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta$, and $\beta - \alpha \le 2\pi$.**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$




**EXAMPLE 8:** Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

**Solution** We sketch the region to determine its boundaries and find the limits of integration . The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and $\theta$ runs from $-\pi/2$ to $\pi/2$. The area is

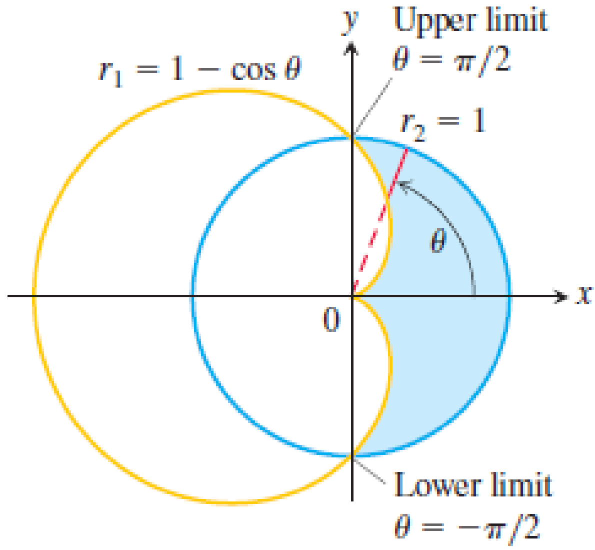
$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Eq. (1)}$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$

$$= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \qquad r_2 = 1 \text{ and } r_1 = 1 - \cos\theta$$

$$= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{0}^{\pi/2} = 2 - \frac{\pi}{4}.$$



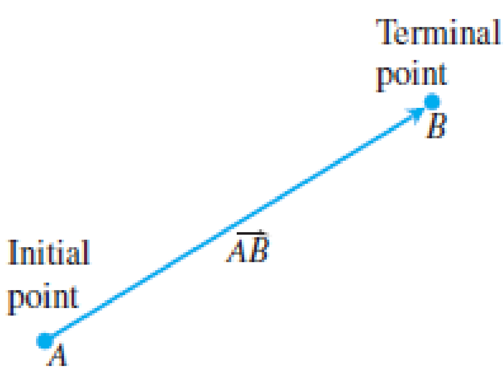
y
Upper limit
$\theta = \pi/2$
$r_1 = 1 - \cos\theta$
$r_2 = 1$
$\theta$
x
0
Lower limit
$\theta = -\pi/2$

## Chapter Two

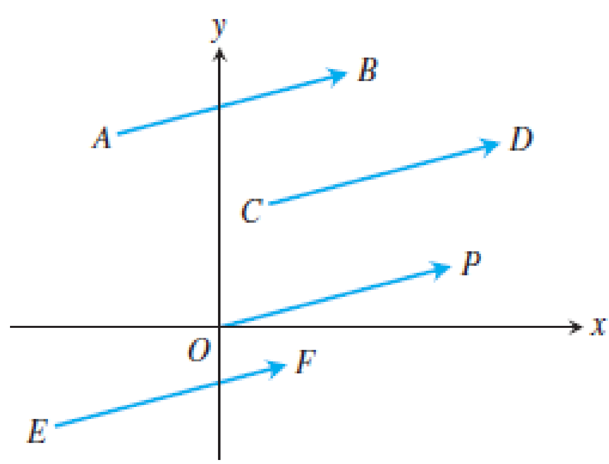
## Vectors

Some of the things we measure are determined simply by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far. To describe a body's velocity, we have to know its direction of motion, as well as how fast it is going. In this section we show how to represent things that have both magnitude and direction in the plane or in space.

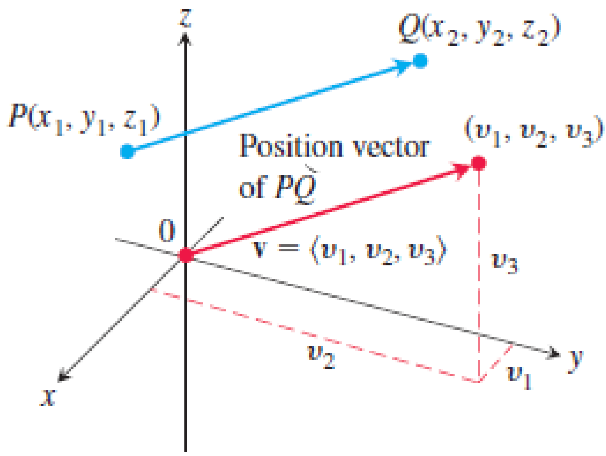
**DEFINITIONS** The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** $A$ and **terminal point** $B$ and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.




The directed line segment $\overrightarrow{AB}$ is called a vector.




The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.




A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and **v** are parallel and have the same length.

**DEFINITION** If **v** is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point $(v_1, v_2)$, then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If **v** is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point $(v_1, v_2, v_3)$, then the **component form** of **v** is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(see Figure 12.10).

**EXAMPLE 1** Find the (**a**) component form and (**b**) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

**Solution**

(**a**) The standard position vector **v** representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(**b**) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

# Vector Algebra Operations

**DEFINITIONS** Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with $k$ a scalar.

**Addition:** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

**Scalar multiplication:** $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

**EXAMPLE 3** Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

**(a)** $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

**Solution**

**(a)** $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

**(b)** $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

**(c)** $\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$ ■

## Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \end{aligned}$$

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

If $\mathbf{v} \neq \mathbf{0}$, then its length $|\mathbf{v}|$ is not zero and

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of the nonzero vector $\mathbf{v}$.

**EXAMPLE 4** Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

**Solution** We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

This unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$. ■

**EXAMPLE 5** If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times its direction of motion.

**Solution** Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underbrace{5}_{\text{Length (speed)}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$ ■

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

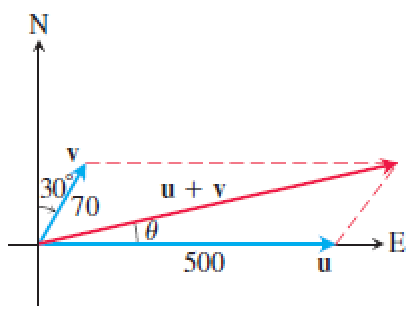
**EXAMPLE 6** A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

**Solution** The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\mathbf{F} = 6\frac{\mathbf{v}}{|\mathbf{v}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3}$$

$$= 6\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}\right).$$

**EXAMPLE 8** A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?




$$\mathbf{u} = \langle 500, 0 \rangle \quad \text{and} \quad \mathbf{v} = \langle 70\cos 60°, 70\sin 60° \rangle = \langle 35, 35\sqrt{3} \rangle.$$

Therefore,

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle = 535\mathbf{i} + 35\sqrt{3}\,\mathbf{j}$$

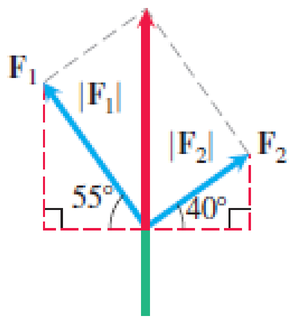
$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

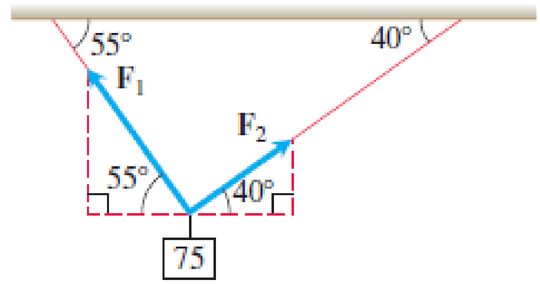
and

$$\theta = \tan^{-1}\frac{35\sqrt{3}}{535} \approx 6.5°.$$

The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east.

**EXAMPLE 9** A 75-N weight is suspended by two wires, Find the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in both wires.




(a)

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75\rangle$

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°,\ |\mathbf{F}_1|\sin 55°\rangle \qquad \text{and} \qquad \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°,\ |\mathbf{F}_2|\sin 40°\rangle.$$

Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75\rangle$, the resultant vector leads to the system of equations

$$-|\mathbf{F}_1|\cos 55° + |\mathbf{F}_2|\cos 40° = 0$$
$$|\mathbf{F}_1|\sin 55° + |\mathbf{F}_2|\sin 40° = 75.$$

Solving for $|\mathbf{F}_2|$ in the first equation and substituting the result into the second equation, we get

$$|\mathbf{F}_2| = \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°} \qquad \text{and} \qquad |\mathbf{F}_1|\sin 55° + \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°}\sin 40° = 75.$$

It follows that

$$|\mathbf{F}_1| = \frac{75}{\sin 55° + \cos 55° \tan 40°} \approx 57.67 \text{ N},$$

and

$$|\mathbf{F}_2| = \frac{75\cos 55°}{\sin 55°\cos 40° + \cos 55°\sin 40°}$$
$$= \frac{75\cos 55°}{\sin(55° + 40°)} \approx 43.18 \text{ N}.$$

The force vectors are then

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°, |\mathbf{F}_1|\sin 55°\rangle \approx \langle -33.08, 47.24\rangle$$

and

$$\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°, |\mathbf{F}_2|\sin 40°\rangle \approx \langle 33.08, 27.76\rangle.$$

# The Dot Product

**DEFINITION** The **dot product** $\mathbf{u} \cdot \mathbf{v}$ (“**u dot v**”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

**EXAMPLE 1** We illustrate the definition.

**(a)** $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$

$$= -6 - 4 + 3 = -7$$

**(b)** $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$

**Dot Product and Angles**

The angle between two nonzero vectors **u** and **v** is $\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$.

The dot product of two vectors **u** and **v** is given by $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$.

**EXAMPLE 2** Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

**Solution** We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians or } 100.98°.$$

The angle formula applies to two-dimensional vectors as well. Note that the angle $\theta$ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$ and obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.

**EXAMPLE 3** Find the angle $\theta$ in the triangle $ABC$ determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$

**Solution** The angle $\theta$ is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are

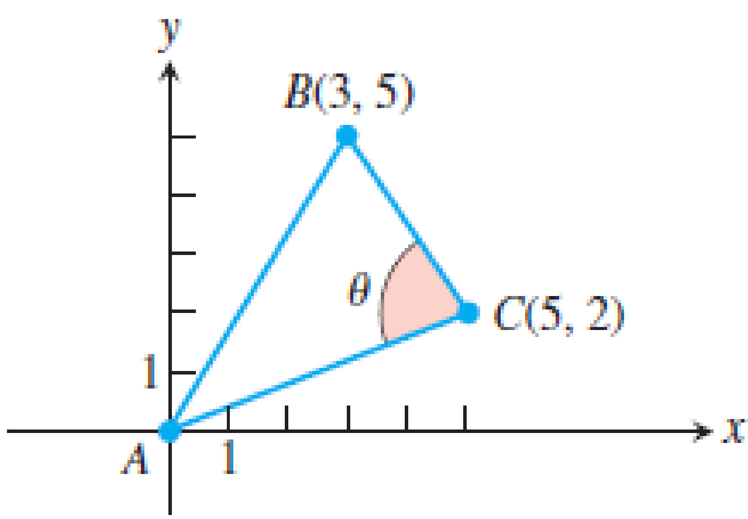
$$\overrightarrow{CA} = \langle -5, -2 \rangle \qquad \text{and} \qquad \overrightarrow{CB} = \langle -2, 3 \rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

$$\begin{aligned} \overrightarrow{CA} \cdot \overrightarrow{CB} &= (-5)(-2) + (-2)(3) = 4 \\ |\overrightarrow{CA}| &= \sqrt{(-5)^2 + (-2)^2} = \sqrt{29} \\ |\overrightarrow{CB}| &= \sqrt{(-2)^2 + (3)^2} = \sqrt{13} \end{aligned}$$

Then applying the angle formula, we have

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right) = \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right) \\ &\approx 78.1^\circ \text{ or } 1.36 \text{ radians.} \end{aligned}$$



## Orthogonal Vectors

**DEFINITION** Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

**EXAMPLE 4** To determine if two vectors are orthogonal, calculate their dot product.

**(a)** $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

**(b)** $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

**(c)** $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) = 0. \end{aligned}$$

## WORK

**DEFINITION** The **work** done by a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

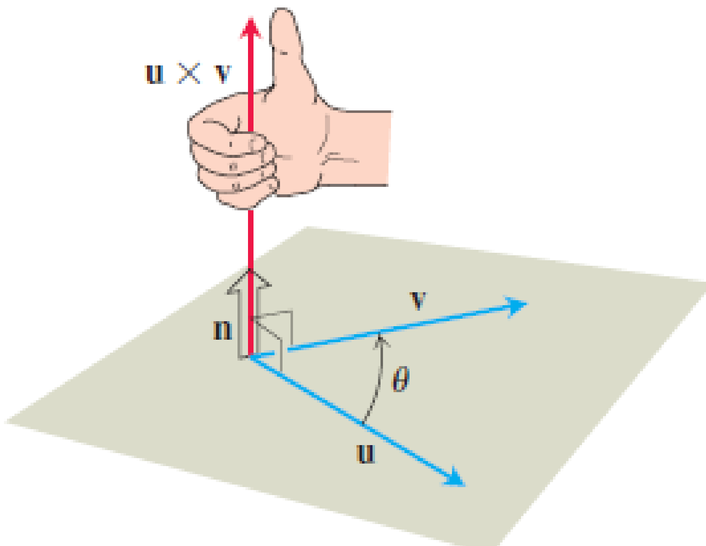
**EXAMPLE 8** If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60°$, the work done by $\mathbf{F}$ in acting from $P$ to $Q$ is

$$\begin{aligned} \text{Work} &= \mathbf{F} \cdot \mathbf{D} && \text{Definition} \\ &= |\mathbf{F}||\mathbf{D}| \cos\theta \\ &= (40)(3)\cos 60° && \text{Given values} \\ &= (120)(1/2) = 60 \text{ J (joules).} \end{aligned}$$

# The Cross Product

**DEFINITION** The cross product $\mathbf{u} \times \mathbf{v}$ ("$\mathbf{u}$ cross $\mathbf{v}$") is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\,\mathbf{n}.$$




The construction of $\mathbf{u} \times \mathbf{v}$.

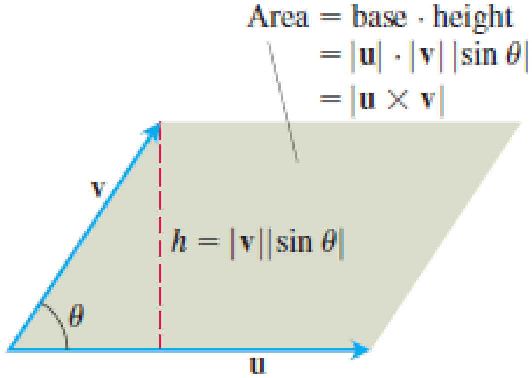
**Parallel Vectors**

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

## $|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$




The parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

### Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

**EXAMPLE 1** Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

**Solution** We expand the symbolic determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \qquad \text{Property 3}$$

**EXAMPLE 2** Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.32).

**Solution** The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

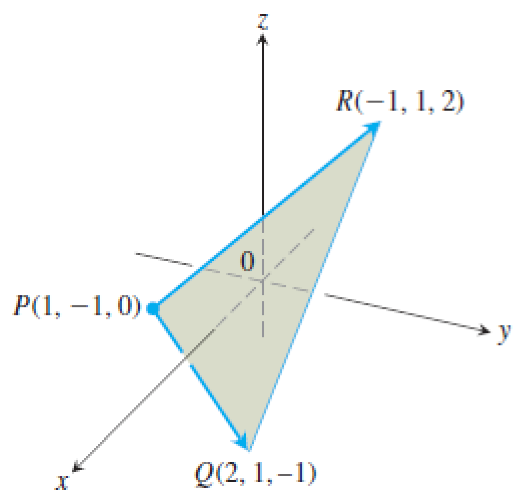
**EXAMPLE 3** Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$

**Solution** The area of the parallelogram determined by $P$, $Q$, and $R$ is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}| \qquad \text{Values from Example 2}$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.




The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane of triangle $PQR$ (Example 2). The area of triangle $PQR$ is half of $|\overrightarrow{PQ} \times \overrightarrow{PR}|$

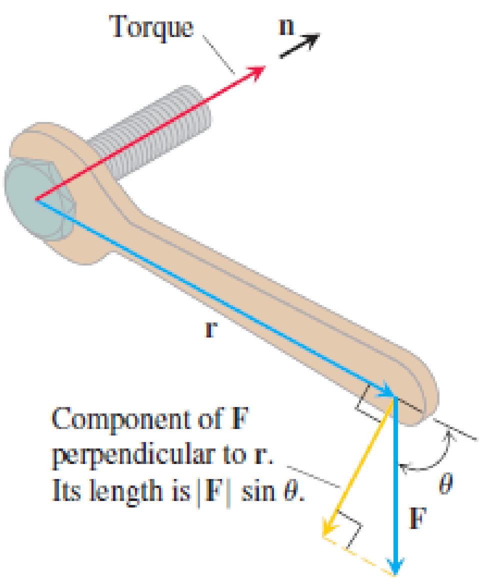
**EXAMPLE 4** Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

**Solution** Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

For ease in calculating the cross product using determinants, we usually write vectors in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ rather than as ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

## Torque




The torque vector describes the tendency of the force F to drive the bolt forward.

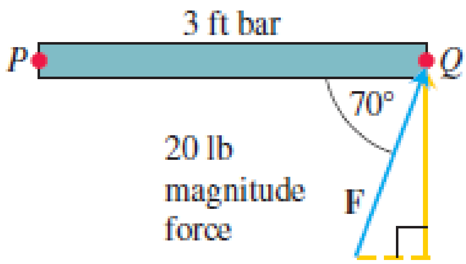
$$\text{Magnitude of torque vector} = |\mathbf{r}||\mathbf{F}|\sin\theta,$$

$$\text{Torque vector} = \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\,\mathbf{n}.$$

**EXAMPLE 5** The magnitude of the torque generated by force **F** at the pivot point *P*

$$|\overrightarrow{PQ} \times \mathbf{F}| = |\overrightarrow{PQ}||\mathbf{F}|\sin 70° \approx (3)(20)(0.94) \approx 56.4 \text{ ft-lb}.$$

In this example the torque vector is pointing out of the page toward you. ■




The magnitude of the
torque exerted by **F** at *P* is about 56.4 ft-lb
The bar rotates counter-
clockwise around *P*.

# Lines and Planes in Space

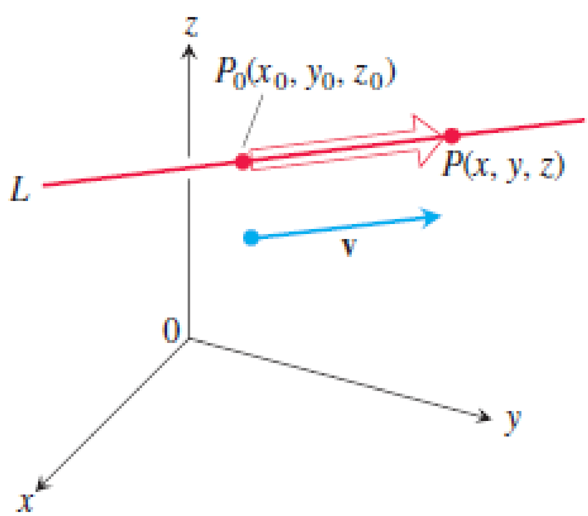
## Lines and Line Segments in Space

**Vector Equation for a Line**

**A vector equation for the line $L$ through $P_0(x_0, y_0, z_0)$ parallel to v is**

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty,$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on $L$ and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.




A point $P$ lies on $L$ through $P_0$ parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.

**Parametric Equations for a Line**

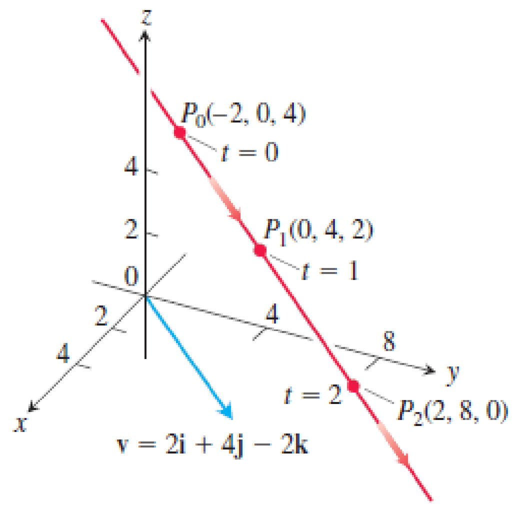
**The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is**

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

**EXAMPLE 1** Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

**Solution** With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$




Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing $t$.

**EXAMPLE 2** Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

**Solution** The vector

$$\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of $t$.

**EXAMPLE 3** Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

**Solution** We begin with equations for the line through $P$ and $Q$, taking them, in this case, from Example 2:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \le t \le 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \le t \le 1.$$

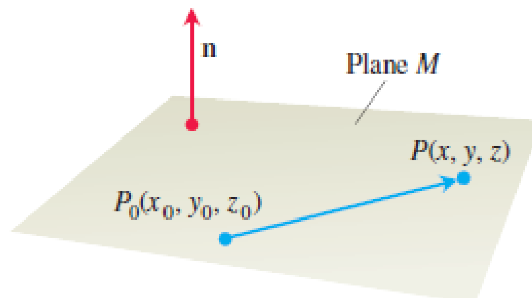
■

# An Equation for a Plane in Space

**Equation for a Plane**

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

| | |
|---|---|
| **Vector equation:** | $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$ |
| **Component equation:** | $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ |
| **Component equation simplified:** | $Ax + By + Cz = D$, where |
| | $D = Ax_0 + By_0 + Cz_0$ |




The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point $P$ lies in the plane through $P_0$ normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

**EXAMPLE 6** Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

**Solution** The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$
$$5x + 2y - z = -22.$$

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of $x$, $y$, and $z$ in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

**EXAMPLE 7** Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

**Solution** We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$
$$3x + 2y + 6z = 6.$$

**EXAMPLE 10** Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

**Solution** The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$t = -1.$$

The point of intersection is

$$(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

## Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors

**EXAMPLE 12** Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

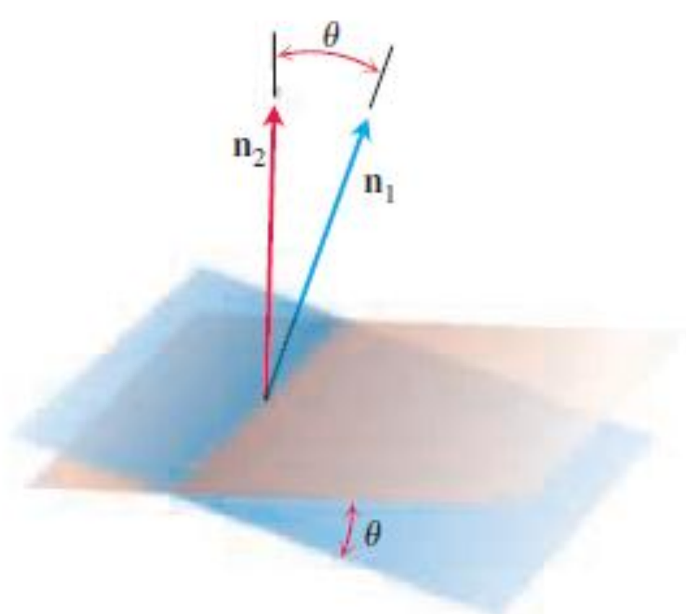
**Solution** The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radians.}$$ About 79 degrees ■




The angle between two planes is obtained from the angle between their normals.

# Chapter Three

# Partial Derivatives

## Partial Derivatives of a Function of Two Variables

If $(x_0, y_0)$ is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.16). This curve is the graph of

the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is $x$; the vertical coordinate is $z$. The $y$-value is held constant at $y_0$, so $y$ is not a variable.

We define the partial derivative of $f$ with respect to $x$ at the point $(x_0, y_0)$ as the ordinary derivative of $f(x, y_0)$ with respect to $x$ at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol $\partial$ rather than the $d$ previously used. In the definition, $h$ represents a real number, positive or negative.

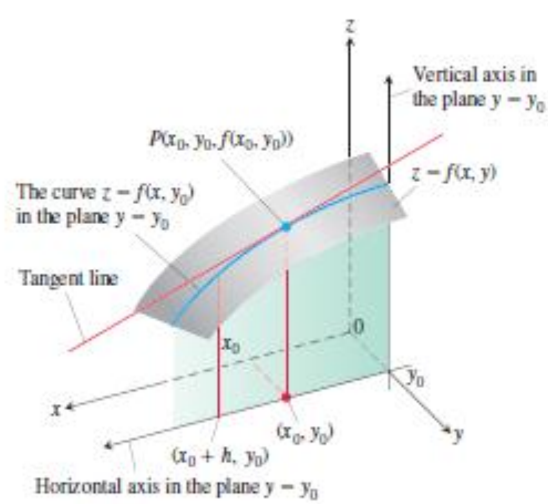



**FIGURE 14.16** The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the $xy$-plane.

**EXAMPLE 1** Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

**Solution** To find $\partial f/\partial x$, we treat $y$ as a constant and differentiate with respect to $x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat $x$ as a constant and differentiate with respect to $y$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

**EXAMPLE 2** Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

**Solution** We treat $x$ as a constant and $f$ as a product of $y$ and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

**EXAMPLE 3** Find $f_x$ and $f_y$ as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

**Solution** We treat $f$ as a quotient. With $y$ held constant, we use the quotient rule to get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

With $x$ held constant and again applying the quotient rule, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

**EXAMPLE 6** If $x$, $y$, and $z$ are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) && \text{x held constant} \\
&= x \cos(y + 3z) \frac{\partial}{\partial z}(y + 3z) && \text{Chain rule} \\
&= 3x \cos(y + 3z). && \text{y held constant}
\end{aligned}$$

■

## Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$\dfrac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to $y$, then with respect to $x$.

$f_{yx} = (f_y)_x$ Means the same thing.

**EXAMPLE 9** If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y\, \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x\, \partial y}.$$

**Solution** The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

**THEOREM 2—The Mixed Derivative Theorem**
If $f(x, y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are defined throughout an open region containing a point $(a, b)$ and are all continuous at $(a, b)$, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

**EXAMPLE 10** Find $\frac{\partial^2 w}{\partial x\, \partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

**Solution** The symbol $\partial^2 w/\partial x\, \partial y$ tells us to differentiate first with respect to $y$ and then with respect to $x$. However, if we interchange the order of differentiation and differentiate first with respect to $x$ we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y\, \partial x} = 1.$$

If we differentiate first with respect to $y$, we obtain $\partial^2 w/\partial x\, \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for $w$ at all points $(x_0, y_0)$. ■

**EXAMPLE 11** Find $f_{yxyz}$ if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

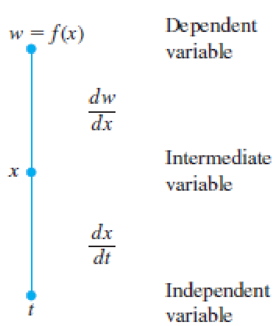
**Solution** We first differentiate with respect to the variable $y$, then $x$, then $y$ again, and finally with respect to $z$:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

## The Chain Rule

To find $dw/dt$, we read down the route from $w$ to $t$, multiplying derivatives along the way.

**Chain Rule**




$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

The Chain Rule for functions of a single variable studied in Section 3.6 says that when $w = f(x)$ is a differentiable function of $x$ and $x = g(t)$ is a differentiable function of $t$, $w$ is a differentiable function of $t$ and $dw/dt$ can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

For this composite function $w(t) = f(g(t))$, we can think of $t$ as the independent variable and $x = g(t)$ as the "intermediate variable," because $t$ determines the value of $x$ which in turn gives the value of $w$ from the function $f$. We display the Chain Rule in a "dependency diagram" in the margin. Such diagrams capture which variables depend on which.

For functions of several variables the Chain Rule has more than one form, which depends on how many independent and intermediate variables are involved. However, once the variables are taken into account, the Chain Rule works in the same way we just discussed.

### Functions of Two Variables

The Chain Rule formula for a differentiable function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of $t$ is given in the following theorem.

**THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables**

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of $t$, then the composition $w = f(x(t), y(t))$ is a differentiable function of $t$ and

$$\frac{dw}{dt} = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

**EXAMPLE 1** Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to $t$ along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

**Solution** We apply the Chain Rule to find $dw/dt$ as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\cdot\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\cdot\frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of $t$,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}(2\cos 2t) = \cos 2t.$$

In either case, at the given value of $t$,

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2\frac{\pi}{2}\right) = \cos\pi = -1.$$

## Functions of Three Variables

You can probably predict the Chain Rule for functions of three intermediate variables, as it only involves adding the expected third term to the two-variable formula.

**THEOREM 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables**

If $w = f(x, y, z)$ is differentiable and $x$, $y$, and $z$ are differentiable functions of $t$, then $w$ is a differentiable function of $t$ and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

**EXAMPLE 2** Find $dw/dt$ if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as $t$ changes. What is the derivative's value at $t = 0$?

**Solution** Using the Chain Rule for three intermediate variables, we have

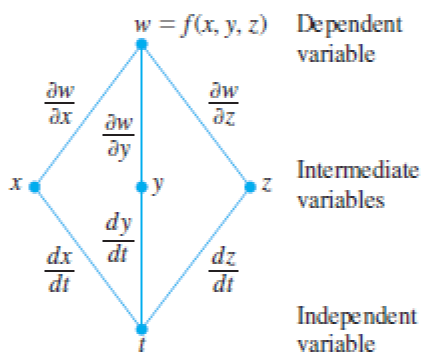
$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}\\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1)\\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for intermediate variables.

so

$$\left.\frac{dw}{dt}\right|_{t=0} = 1 + \cos(0) = 2.$$

**Chain Rule**




$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

## Functions Defined on Surfaces

If we are interested in the temperature $w = f(x, y, z)$ at points $(x, y, z)$ on the earth's surface, we might prefer to think of $x$, $y$, and $z$ as functions of the variables $r$ and $s$ that give the points' longitudes and latitudes. If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of $r$ and $s$ with the composite function

$$w = f(g(r, s), h(r, s), k(r, s)).$$

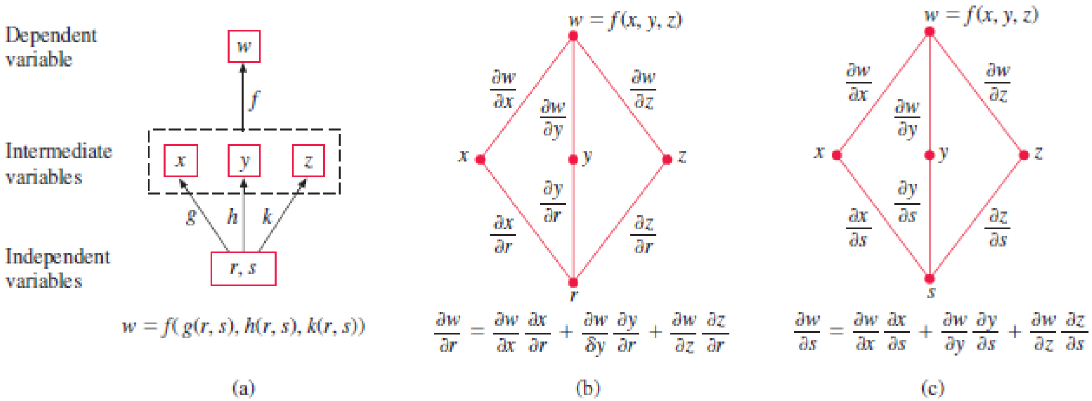
Under the conditions stated below, $w$ has partial derivatives with respect to both $r$ and $s$ that can be calculated in the following way.

**THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables**

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then $w$ has partial derivatives with respect to $r$ and $s$, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$




**EXAMPLE 3** Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of $r$ and $s$ if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

**Solution** Using the formulas in Theorem 7, we find

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

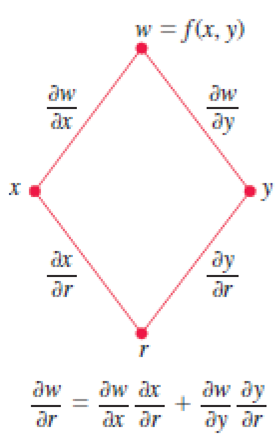
$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$ Substitute for intermediate variable $z$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.$$

If $f$ is a function of two intermediate variables instead of three, each equation in Theorem 7 becomes correspondingly one term shorter.

**Chain Rule**



**THEOREM 8—A Formula for Implicit Differentiation**
Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines $y$ as a differentiable function of $x$. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \qquad (1)$$

**EXAMPLE 5** Use Theorem 8 to find $dy/dx$ if $y^2 - x^2 - \sin xy = 0$.

**Solution** Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

This calculation is significantly shorter than a single-variable calculation using implicit differentiation. ■

**EXAMPLE 6** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

**Solution** Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad \text{and} \qquad F_z = 2z + xye^{xz} + \cos y.$$

Since $F(0, 0, 0) = 0$, $F_z(0, 0, 0) = 1 \neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x, y, z) = 0$ defines $z$ as a differentiable function of $x$ and $y$ near the point $(0, 0, 0)$. From Equations (2),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$ ■

# Directional Derivatives and Gradient Vectors

## Directional Derivatives in the Plane

We know from Section 14.4 that if $f(x, y)$ is differentiable, then the rate at which $f$ changes with respect to $t$ along a differentiable curve $x = g(t)$, $y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation gives the rate of change of $f$ with respect to increasing $t$ and therefore depends, among other things, on the direction of motion along the curve. If the curve is a straight line and $t$ is the arc length parameter along the line measured from $P_0$ in the direction of a given unit vector $\mathbf{u}$, then $df/dt$ is the rate of change of $f$ with respect to distance in its domain in the direction of $\mathbf{u}$. By varying $\mathbf{u}$, we find the rates at which $f$ changes with respect to distance as we move through $P_0$ in different directions. We now define this idea more precisely.

The **directional derivative** defined by Equation (1) is also denoted by

$$D_{\mathbf{u}}f(P_0) \qquad \text{or} \qquad D_{\mathbf{u}}f\big|_{P_0}$$

"The derivative of $f$ in the direction of $\mathbf{u}$, evaluated at $P_0$"

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of $f$ at $P_0$ in the $\mathbf{i}$ and $\mathbf{j}$ directions. This observation can be seen by comparing Equation (1) to the definitions of the two partial derivatives given in Section 14.3.

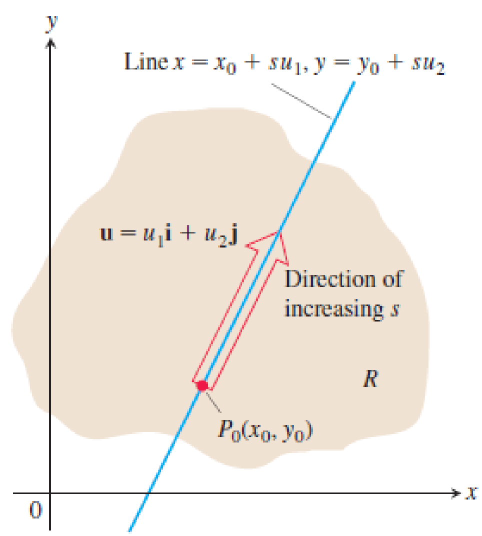



**FIGURE 14.27** The rate of change of $f$ in the direction of $\mathbf{u}$ at a point $P_0$ is the rate at which $f$ changes along this line at $P_0$.

**DEFINITION** The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f\big|_{P_0} \qquad \text{or} \qquad \nabla f(x_0, y_0).$$

The notation $\nabla f$ is read "grad $f$" as well as "gradient of $f$" and "del $f$." The symbol $\nabla$ by itself is read "del." Another notation for the gradient is grad $f$. Using the gradient notation, we restate Equation (3) as a theorem.

**THEOREM 9—The Directional Derivative Is a Dot Product**
If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \nabla f|_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient $\nabla f$ at $P_0$ with the vector $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

**EXAMPLE 2** Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

**Solution** Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of $f$ are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))\Big|_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))\Big|_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

The gradient of $f$ at $(2, 0)$ is

x

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.29). The derivative of $f$ at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$\begin{aligned} D_{\mathbf{u}} f|_{(2,0)} &= \nabla f|_{(2,0)} \cdot \mathbf{u} && \text{Eq. (4) with the } D_{\mathbf{u}} f|_{P_0} \text{ notation} \\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1. \end{aligned}$$

■

Evaluating the dot product in the brief version of Equation (4) gives

t

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where $\theta$ is the angle between the vectors $\mathbf{u}$ and $\nabla f$, and reveals the following properties.

**Properties of the Directional Derivative $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$**

1. The function $f$ increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of $\nabla f$. That is, at each point $P$ in its domain, $f$ increases most rapidly in the direction of the gradient vector $\nabla f$ at $P$. The derivative in this direction is
$$D_{\mathbf{u}} f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, $f$ decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in $f$ because $\theta$ then equals $\pi/2$ and
$$D_{\mathbf{u}} f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

**EXAMPLE 3** Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

**(a)** increases most rapidly at the point (1, 1), and

**(b)** decreases most rapidly at (1, 1).

**(c)** What are the directions of zero change in $f$ at (1, 1)?

**Solution**

**(a)** The function increases most rapidly in the direction of $\nabla f$ at (1, 1). The gradient there is
$$\nabla f|_{(1,1)} = (x\mathbf{i} + y\mathbf{j})\Big|_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is
$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(b)** The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is
$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(c)** The directions of zero change at (1, 1) are the directions orthogonal to $\nabla f$:
$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

## Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

so the properties listed earlier for functions of two variables extend to three variables. At any given point, $f$ increases most rapidly in the direction of $\nabla f$ and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to $\nabla f$, the derivative is zero.

## EXAMPLE 6

**(a)** Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

**(b)** In what directions does $f$ change most rapidly at $P_0$, and what are the rates of change in these directions?

### Solution

**(a)** The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of $f$ at $P_0$ are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of $f$ at $P_0$ is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of $f$ at $P_0$ in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f|_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

**(b)** The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

# Tangent Planes and Differentials

**DEFINITIONS** The **tangent plane** to the level surface $f(x, y, z) = c$ of a differentiable function $f$ at a point $P_0$ where the gradient is not zero is the plane through $P_0$ normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at $P_0$ is the line through $P_0$ parallel to $\nabla f|_{P_0}$.

The results of Section 12.5 imply that the tangent plane and normal line satisfy the following equations, as long as the gradient at the point $P_0$ is not the zero vector.

**Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$**

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \quad (1)$$

**Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$**

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \quad (2)$$

**EXAMPLE 1** Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

**Solution** The surface is shown in Figure 14.34.

The tangent plane is the plane through $P_0$ perpendicular to the gradient of $f$ at $P_0$. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at $P_0$ is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The partial derivatives of $F$ are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

The formula

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

for the plane tangent to the level surface at $P_0$ therefore reduces to

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0,$$

as long as the gradient is not the zero vector at the point $P_0$.

## Estimating Change in a Specific Direction

The directional derivative plays a role similar to that of an ordinary derivative when we want to estimate how much the value of a function $f$ changes if we move a small distance $ds$ from a point $P_0$ to another point nearby. If $f$ were a function of a single variable, we would have

$$df = f'(P_0)\,ds. \qquad \text{Ordinary derivative} \times \text{increment}$$

For a function of two or more variables, we use the formula

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})\,ds, \qquad \text{Directional derivative} \times \text{increment}$$

where $\mathbf{u}$ is the direction of the motion away from $P_0$.

> **Estimating the Change in $f$ in a Direction u**
> To estimate the change in the value of a differentiable function $f$ when we move a small distance $ds$ from a point $P_0$ in a particular direction $\mathbf{u}$, use the formula
>
> $$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional}\\ \text{derivative}}} \underbrace{ds}_{\substack{\text{Distance}\\ \text{increment}}}$$

**EXAMPLE 4** Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

**Solution** We first find the derivative of $f$ at $P_0$ in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of $f$ at $P_0$ is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})\Big|_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change $df$ in $f$ that results from moving $ds = 0.1$ unit away from $P_0$ in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit}.$$

# Chapter Four

## MULTIPLE INTEGRALS

### Double Integrals over Rectangles

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region $R$,

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

**THEOREM 1 Fubini's Theorem (First Form)**

If $f(x, y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

**EXAMPLE 1** Evaluating a Double Integral

Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 1 - 6x^2 y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

**Solution** By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_0^2 (1 - 6x^2 y)\, dx\, dy = \int_{-1}^{1} \left[x - 2x^3 y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (2 - 16y)\, dy = \left[2y - 8y^2\right]_{-1}^{1} = 4.$$

Reversing the order of integration gives the same answer:

$$\int_0^2 \int_{-1}^{1} (1 - 6x^2 y)\, dy\, dx = \int_0^2 \left[y - 3x^2 y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 \left[(1 - 3x^2) - (-1 - 3x^2)\right] dx$$

$$= \int_0^2 2\, dx = 4.$$

**EXAMPLE 2** Finding Volume

Find the volume of the prism whose base is the triangle in the $xy$-plane bounded by the $x$-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

**Solution** See Figure 15.11 on page 1075. For any $x$ between 0 and 1, $y$ may vary from $y = 0$ to $y = x$ (Figure 15.11b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

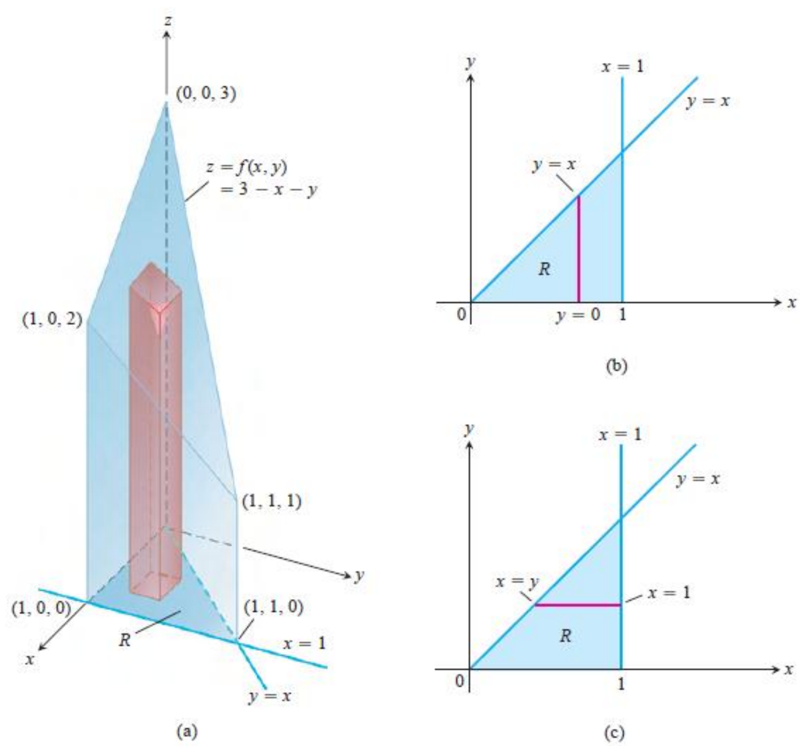
$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.11c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

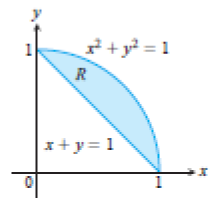


## Finding Limits of Integration

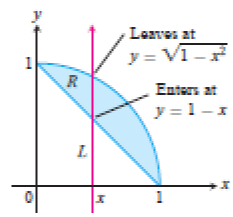
We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating $\iint_R f(x, y)\,dA$, integrating first with respect to $y$ and then with respect to $x$, do the following:

1. *Sketch*. Sketch the region of integration and label the bounding curves.



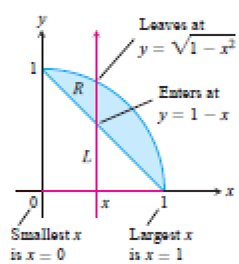

2. *Find the y-limits of integration*. Imagine a vertical line $L$ cutting through $R$ in the direction of increasing $y$. Mark the $y$-values where $L$ enters and leaves. These are the $y$-limits of integration and are usually functions of $x$ (instead of constants).




3. *Find the x-limits of integration*. Choose $x$-limits that include all the vertical lines through $R$. The integral shown here is

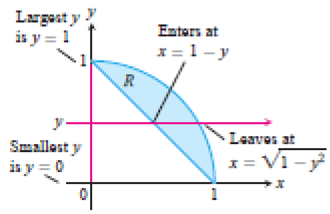
$$\iint_R f(x, y)\, dA =$$

$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$



To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$




**EXAMPLE 4** Reversing the Order of Integration

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.

**Solution** The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 15.13a).

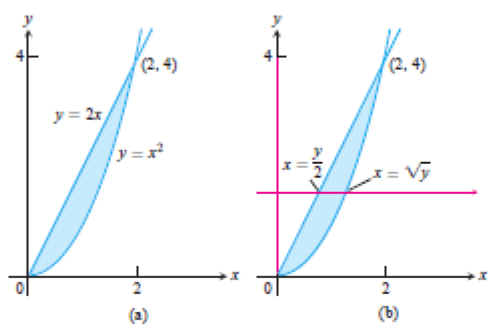



FIGURE 15.13 Region of integration for Example 4.

**DEFINITION Area**

The area of a closed, bounded plane region $R$ is

$$A = \iint_R dA.$$

**EXAMPLE 1** Finding Area

Find the area of the region $R$ bounded by $y = x$ and $y = x^2$ in the first quadrant.

**Solution** We sketch the region (Figure 15.15), noting where the two curves intersect, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[ y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Notice that the single integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.5. ■

**EXAMPLE 2** Finding Area

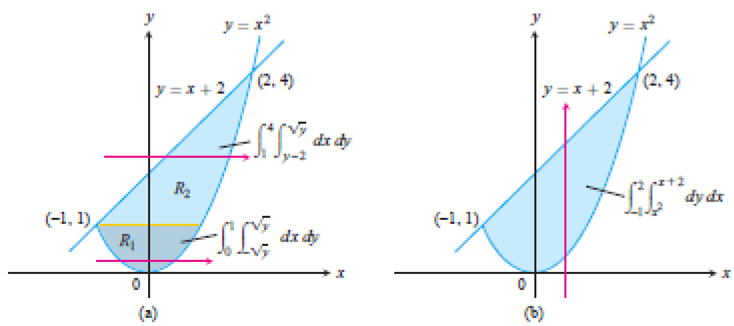
Find the area of the region $R$ enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

**Solution** If we divide $R$ into the regions $R_1$ and $R_2$ shown in Figure 15.16a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$

On the other hand, reversing the order of integration (Figure 15.16b) gives

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$




This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

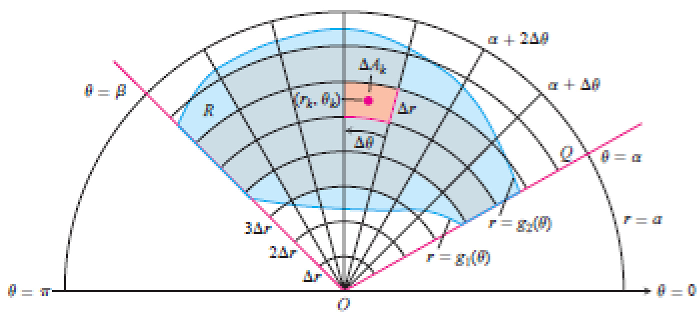
$$A = \int_{-1}^{2} \Big[ y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}.$$ ■

## Double Integrals in Polar Form

### Integrals in Polar Coordinates

When we defined the double integral of a function over a region $R$ in the $xy$-plane, we began by cutting $R$ into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant $x$-values or constant $y$-values. In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant $r$- and $\theta$-values.

Suppose that a function $f(r, \theta)$ is defined over a region $R$ that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of $\theta$ between $\alpha$ and $\beta$. Then $R$ lies in a fan-shaped region $Q$ defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$. See Figure 15.21.



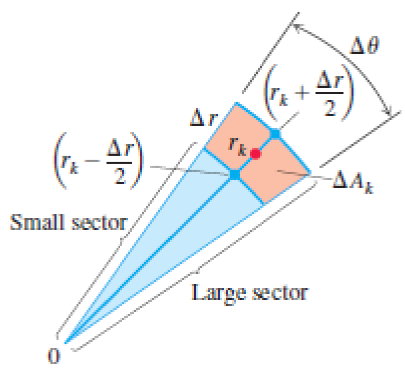




**FIGURE 15.23** The observation that

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix}$$

leads to the formula $\Delta A_k = r_k\, \Delta r\, \Delta\theta$.

We cover $Q$ by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, 2\Delta r, \ldots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \qquad \theta = \alpha + \Delta\theta, \qquad \theta = \alpha + 2\Delta\theta, \qquad \ldots, \qquad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition $Q$ into small patches called "polar rectangles."

We number the polar rectangles that lie inside $R$ (the order does not matter), calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We let $(r_k, \theta_k)$ be any point in the polar rectangle whose area is $\Delta A_k$. We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, \Delta A_k.$$

If $f$ is continuous throughout $R$, this sum will approach a limit as we refine the grid to make $\Delta r$ and $\Delta\theta$ go to zero. The limit is called the double integral of $f$ over $R$. In symbols,

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, dA.$$

To evaluate this limit, we first have to write the sum $S_n$ in a way that expresses $\Delta A_k$ in terms of $\Delta r$ and $\Delta\theta$. For convenience we choose $r_k$ to be the average of the radii of the inner and outer arcs bounding the $k$th polar rectangle $\Delta A_k$. The radius of the inner arc bounding $\Delta A_k$ is then $r_k - (\Delta r/2)$ (Figure 15.23). The radius of the outer arc is $r_k + (\Delta r/2)$.

The area of a wedge-shaped sector of a circle having radius $r$ and angle $\theta$ is

$$A = \frac{1}{2}\theta \cdot r^2,$$

as can be seen by multiplying $\pi r^2$, the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius:} \qquad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$$

$$\text{Outer radius:} \qquad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta.$$

Therefore,

$$\begin{aligned}\Delta A_k &= \text{area of large sector} - \text{area of small sector}\\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\,\Delta r) = r_k\,\Delta r\,\Delta\theta.\end{aligned}$$

Combining this result with the sum defining $S_n$ gives

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta.$$

As $n \to \infty$ and the values of $\Delta r$ and $\Delta\theta$ approach zero, these sums converge to the double integral

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, r\, dr\, d\theta.$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to $r$ and $\theta$ as

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

## Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. We illustrate this using the region $R$ shown in Figure 15.24. To evaluate $\iint_R f(r, \theta)\, dA$ in polar coordinates, integrating first with respect to $r$ and then with respect to $\theta$, take the following steps.

1. *Sketch*. Sketch the region and label the bounding curves (Figure 15.24a).
2. *Find the r-limits of integration*. Imagine a ray $L$ from the origin cutting through $R$ in the direction of increasing $r$. Mark the $r$-values where $L$ enters and leaves $R$. These are the $r$-limits of integration. They usually depend on the angle $\theta$ that $L$ makes with the positive $x$-axis (Figure 15.24b).
3. *Find the θ-limits of integration*. Find the smallest and largest $\theta$-values that bound $R$. These are the $\theta$-limits of integration (Figure 15.24c). The polar iterated integral is

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r, \theta)\, r\, dr\, d\theta.$$

**EXAMPLE 1** Find the limits of integration for integrating $f(r, \theta)$ over the region $R$ that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

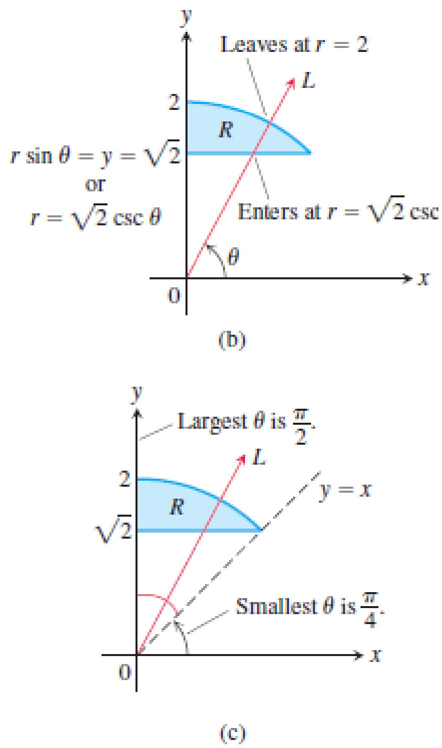



**FIGURE 15.24** Finding the limits of integration in polar coordinates.

## Solution

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters $R$ where $r = 1$ and leaves where $r = 1 + \cos\theta$.

3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect $R$ run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

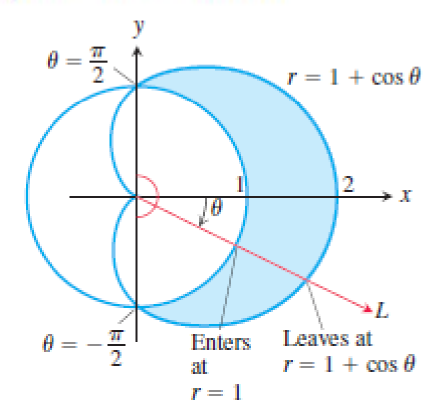



**FIGURE 15.25** Finding the limits of integration in polar coordinates for the region in Example 1.

### Area in Polar Coordinates

The area of a closed and bounded region $R$ in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

**EXAMPLE 2** Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

**Solution** We graph the lemniscate to determine the limits of integration (Figure 15.26) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

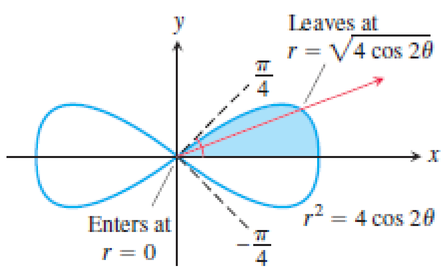



**FIGURE 15.26** To integrate over the shaded region, we run $r$ from 0 to $\sqrt{4\cos 2\theta}$ and $\theta$ from 0 to $\pi/4$ (Example 2).

## Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y)\,dx\,dy$ into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of $R$. The Cartesian integral then becomes

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta,$$

where $G$ denotes the same region of integration, but now described in polar coordinates. This is like the substitution method in Chapter 5 except that there are now two variables to substitute for instead of one. Notice that the area differential $dx\,dy$ is not replaced by $dr\,d\theta$ but by $r\,dr\,d\theta$. A more general discussion of changes of variables (substitutions) in multiple integrals is given in Section 15.8.

**EXAMPLE 3** Evaluate

$$\iint_R e^{x^2+y^2}\,dy\,dx,$$

where $R$ is the semicircular region bounded by the $x$-axis and the curve $y = \sqrt{1 - x^2}$ (Figure 15.27).

**Solution** In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either $x$ or $y$. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need

to evaluate it. Polar coordinates make this possible. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ give

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^{1} e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1).$$

The $r$ in the $r\,dr\,d\theta$ is what allowed us to integrate $e^{r^2}$. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■

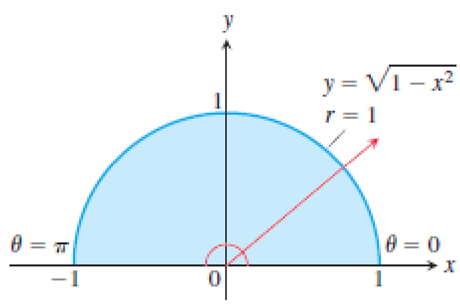



**FIGURE 15.27** The semicircular region in Example 3 is the region

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi.$$

to evaluate it. Polar coordinates make this possible. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ give

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e - 1)\,d\theta = \frac{\pi}{2}(e - 1).$$

The $r$ in the $r\,dr\,d\theta$ is what allowed us to integrate $e^{r^2}$. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■

**EXAMPLE 4** Evaluate the integral

$$\int_0^1\int_0^{\sqrt{1-x^2}}(x^2 + y^2)\,dy\,dx.$$

**Solution** Integration with respect to $y$ gives

$$\int_0^1\left(x^2\sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3}\right)dx,$$

which is difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure 15.27, first quadrant.) Substituting the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le \pi/2$, and $0 \le r \le 1$, and replacing $dy\,dx$ by $r\,dr\,d\theta$ in the double integral, we get

$$\int_0^1\int_0^{\sqrt{1-x^2}}(x^2 + y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^1 (r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{\pi}{8}.$$

The polar coordinate transformation is effective here because $x^2 + y^2$ simplifies to $r^2$ and the limits of integration become constants. ■

**EXAMPLE 5** Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the $xy$-plane.

**Solution** The region of integration $R$ is bounded by the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1, 0 \leq \theta \leq 2\pi$. The solid region is shown in Figure 15.28. The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta \qquad r^2 = x^2 + y^2,\quad dA = r\, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

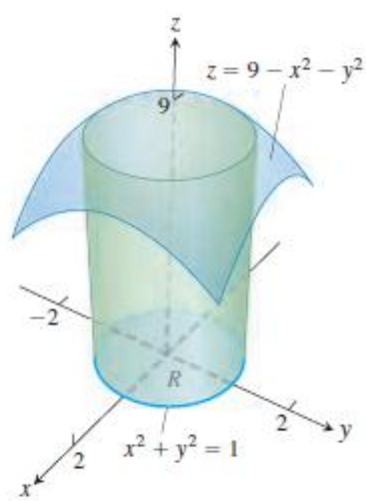



**FIGURE 15.28** The solid region in Example 5.

**EXAMPLE 6** Using polar integration, find the area of the region $R$ in the $xy$-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

**Solution** A sketch of the region $R$ is shown in Figure 15.29. First we note that the line $y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next we observe that the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$. Moreover, the radial line from the origin through the point $(\sqrt{3}, 1)$ has slope $1/\sqrt{3} = \tan\theta$, giving its angle of inclination as $\theta = \pi/6$. This information is shown in Figure 15.29.

Now, for the region $R$, as $\theta$ varies from $\pi/6$ to $\pi/3$, the polar coordinate $r$ varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r\sin\theta$ for $y$ in the equation for the horizontal line, we have $r\sin\theta = 1$, or $r = \csc\theta$, which is the polar equation of the line. The polar equation for the circle is $r = 2$. So in polar coordinates, for $\pi/6 \le \theta \le \pi/3$, $r$ varies from $r = \csc\theta$ to $r = 2$. It follows that the iterated integral for the area is

$$\begin{aligned}
\iint_R dA &= \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta \\
&= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta \\
&= \int_{\pi/6}^{\pi/3}\frac{1}{2}\left[4 - \csc^2\theta\right] d\theta \\
&= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3} \\
&= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}.
\end{aligned}$$

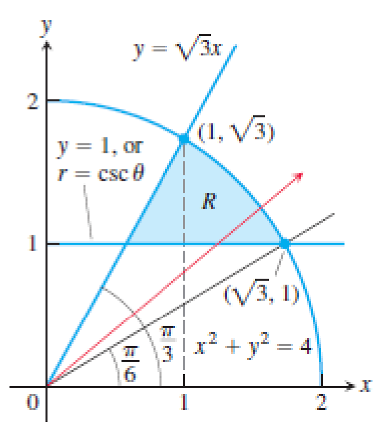



**FIGURE 15.29** The region $R$ in Example 6.

# Triple Integrals

**DEFINITION** The **volume** of a closed, bounded region $D$ in space is

$$V = \iiint_D dV.$$

Example 1:

$$\int\int\int dx\, dy\, dz = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=0}^{4-x^2} dz\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}(4-x^2)dy\, dx$$

$$= \int_{x=0}^{2}(4-x^2)y\Big|_{y=0}^{6} dx$$

$$= \int_{x=0}^{2}(2-6x^2)\, dx = 32$$

Example 2

$$\int_{-1}^{1}\int_{0}^{1}\int_{0}^{2}(x+y+z)\, dy\, dx\, dz =$$

$$\int_{-1}^{1}\int_{0}^{1} = \left[xy+\tfrac{1}{2}y^2+zy\right]_0^2 dx\, dz$$

$$= \int_{-1}^{1}\int_{0}^{1}(2x+2+2z)\, dx\, dz$$

$$= \int_{-1}^{1}\left[x^2+2x+2zx\right]_0^1 dz$$

$$= \int_{-1}^{1}(3+2z)\, dz = \left[3z+z^2\right]_{-1}^{1} = 6$$

Example 3

$$\iiint_S 2x^3y^2z\,dV = \int_0^1 \int_{x^2}^{x} \int_{x-y}^{x+y} 2x^3y^2z\,dz\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} \left\{ x^3y^2z^2 \Big|_{x-y}^{x+y} \right\} dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} x^3y^2[(x+y)^2 - (x-y)^2]\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} 4x^4y^3\,dy\,dx = \int_0^1 \left\{ x^4y^4 \Big|_{x^2}^{x} \right\} dx$$

$$= \int_0^1 (x^8 - x^{12})\,dx = \frac{1}{9} - \frac{1}{13} = \frac{4}{117}. \blacksquare$$