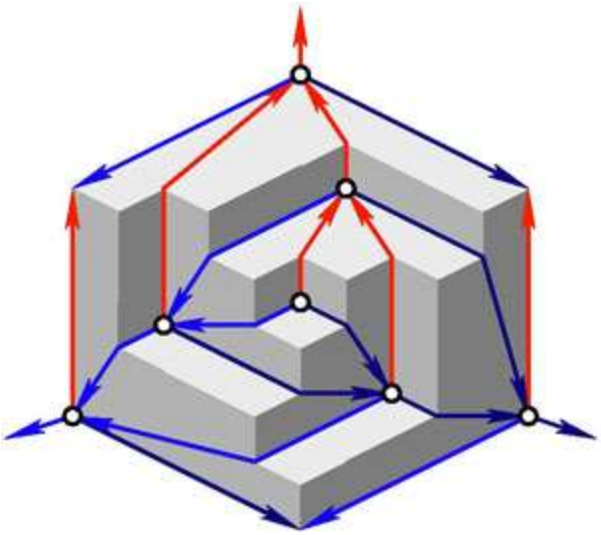
College of Computer Science and Information Technology

Computer Science Department



DISCRETE STRUCTURE

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DISCRETE STRUCTURE (COMP 104)

WHAT IS DISRECT MATHEMATICS

It is the part of mathematics devoted to the stud of discrete objects (unconnected) elements.

There are several important reasons for studying discrete mathematics:

1. We can develop our mathematical ability
2. Discrete mathematic is the gateway to more advanced courses in all part of math.
3. Discrete mathematics provides the math foundations for many computer science courses
4. Discrete mathematics contains the necessary math back ground for solving problems in operation research, chemistry, engineering….

Topics:

- Sets, Subsets
- Operations on sets
- Sequences , Properties of Integers
- Matrices
- Propositional and Logical Operations , Conditional Statements
- Mathematical Induction
- Product sets and Partitions
- Relations and Diagraphs , Paths in Relations and Digraphs
- Properties of Relations , Equivalence Relations
- Computer Representation of Relations and Digraphs
- Operations and Relations
- Functions , Functions for Computer Science
- Growth of Functions, Permutation Functions
- Trees, Labeled Trees
- Tree Searching, Undirected Trees
- Graph, Common graphs
- Some important concepts
- Kinds of graphs, More graphs

CHAPTER ONE

- Sets
- Subsets
- Operations on sets
- Computer Representation of Sets
- Cartesian product

- **Sets**

The word set is used in mathematics to mean any well-defined collection of items. The items in a set are called the **elements** of the set.

For example, we can refer to the set of all the employees of a particular Company or the set of all the integers that are divisible by 5.

A specific set can be **defined in two ways**:

- If there are only a **few** elements, they can be listed individually, by writing them between braces ('curly' brackets) and placing commas in between. For example, the set of positive odd numbers less than 10 can be written in the following way: {1, 3, 5, 7, 9}

  If there is a **clear** pattern to the elements, an ellipsis (three dots) can be used. For example, the set of odd numbers between 0 and 50 can be written: {1, 3, 5, 7, ..., 49}

  Some **infinite** sets can also be written in this way; for example, the set of all positive odd numbers can be written: {1, 3, 5, 7, ...}

- The second way of writing down a set is to use a **property** that defines the elements of the set. Braces are used with this notation also.

  For example, the set of odd numbers between 0 and 50 can be written: {x: x is odd and $0 < x < 50$}

  The colon is read 'such that', so the definition reads 'the set of all x such that x is odd and $0 < x < 50$'.

  - The most **important** of these for our subsequent work are listed below:

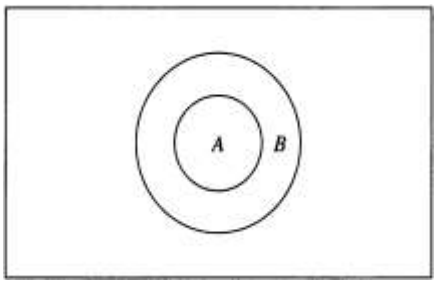
  N is the set of **natural** numbers (or positive integers): {1, 2, 3, 4, ...}.

  J is the set of **integers**: {..., –3, –2, –1, 0, 1, 2, 3, ...}.

  Q is the set of **rational** numbers: {x: x = m/n for some integers m and n}

- Another way is to use **venn diagrams**. Venn diagrams can be used to illustrate that a set A is a subset of a set B. We draw the universal set U as a rectangle. Within this rectangle we draw a circle for B. Because A is a subset of B, we draw the circle for A within the circle for B. This relationship is shown in follows Figure.

  $A \subseteq B \leftrightarrow \forall X (X \in A \rightarrow X \in B)$ is True.



Venn Diagram Showing that A is a Subset of B.

Types of set

- **Subsets**

Definition// Let *A* and *B* be sets.We say that *B* is a ***subset*** of *A*, and write $B \subseteq A$, if every element of *B* is an element of *A*.

For example, let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 4\}$, and $C = \{2, 4, 6\}$. Then *B_A*, but *C* is not a subset of *A*, because $6 \subseteq C$ but $6 \subseteq A$.

Definition// Two sets *A* and *B* are ***equal*** if $A \subseteq B$ and $B \subseteq A$.

In other words, $A = B$ if every element of *A* is an element of *B*, and every element of *B* is an element of *A*.

A less formal way of expressing this is:

'Two sets are equal if they have the same elements and the order does **not** matter.

Definition// The **empty set** or **null set** denoted by φ,{ } it is a subset of any set

Just as zero is a very important natural number, the empty set is basic to set theory.

Definition// **singleton** set : a set with **one** element. {φ}

The single element of the the set {φ} is empty set φ

Definition// **finite set** : It is the set which contain finite elements n which we can count.

Definition// **power set** denoted P(s). It is the set of all subsets of the set

EX// what is the power set of {0,1,2}
P({0,1,2})= {φ,{0},{1},{2},{0,1},{0,2},{1,2},{0,1,2}}

#P({0})= {φ,{0}}
If S has n element $p(s)=2^n$

Definition// 2 sets are **disjoint** if their intersection is empty.
EX//
E ={X/ X is even}
O ={X/X is odd}

- **Set Operations**

Definition// Let A and B be sets. The **union** of the sets A and B , denoted by A U B , is the set that contains those elements that are either in A or in B , or in both.
An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B . This tells us that $A \cup B = \{x \mid x \in A \vee x \in B\}$.
The Venn diagram shown in Figure 1 represents the union of two sets A and B . The area that represents A U B is the shaded area within either the circle representing A or the circle representing B .

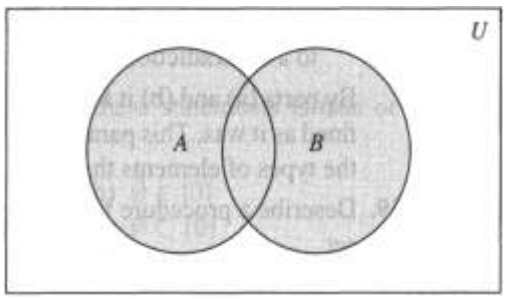


figure 1 Venn Diagram Representing the Union of A and B (A U B is shaded)

EX: The union of the sets { l , 3 , 5 } and { l , 2, 3 } is the set { l , 2 , 3 , 5 } ;
that is, { l , 3 , 5 } U { l , 2, 3 } = { l , 2, 3 , 5 } .

Definition// Let A and B be sets. The **intersection** of the sets A and B , denoted by $A \cap B$, is the set containing those elements in both A and B .
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B . This tells us that
$A \cap B = \{x \mid x \in A \wedge x \in B \}$.
The Venn diagram shown in Figure 2 represents the intersection of two sets A and B . The shaded area that is within both the circles representing the sets A and B is the area that represents the intersection of A and B .

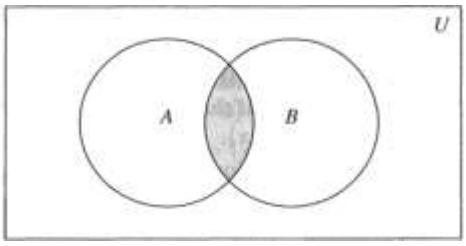


figure 2 Venn Diagram Representing the Intersection of A and B ($A \cap B$ is shaded)

EX: The intersection of the sets { 1 , 3 , 5 } and { 1 , 2 , 3 } is the set { 1 , 3 } ;
that is, { 1 , 3 , 5 } ∩ { l , 2, 3 } ={ 1 , 3 } .

Definition// Let A and B be sets. The **difference** of A and B , denoted by A - B , is the set containing those elements that are in A but not in B . The difference of A and B is also called the **complement** of B with respect to A.
An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$.
This tells us That $A - B = \{x \mid x \in A \wedge x \notin B \}$.

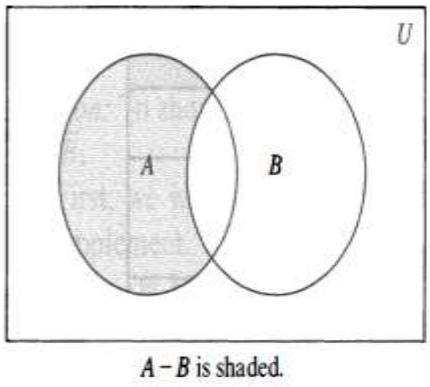
The **Venn diagram** shown in Figure 3 represents the difference of the sets A and B . The shaded area inside the circle that represents A and outside the circle that represents B is the area that represents A - B .

EX: The difference of { 1 , 3 , 5 } and { 1 , 2 , 3 } is the set { 5 } ; that is, { l , 3 , 5 } - { l , 2 , 3 } = { 5 } . This is different from the difference of { 1 , 2 , 3} and { 1 , 3 , 5 } , which is the set {2}.

Definition// Once the universal set U has been specified, the **complement** of a set can be defined. Let U be the universal set. The complement of the set A , denoted by Ã, is the complement of A with respect to U. In other words, the complement of the set A is U -A .
An element belongs to Ã if and only if $x \notin A$. This tells us that if $\tilde{A} = \{x / x \notin A\}$.

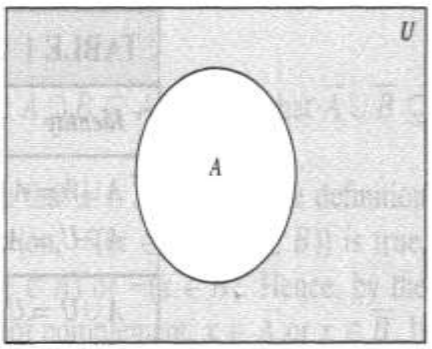
EX: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then Ã = {l, 2, 3 , 4 , 5 , 6 , 7 , 8 , 9 , l0}.

In Figure 4 the shaded area outside the circle representing A i s the area representing Ã.



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of *A* and *B*.



$\bar{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set *A*.

Set Identities

Table 1 lists the most important set identities. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem.

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

- **Prove that** ($\overline{A \cap B} = \overline{A} \cup \overline{B}$.)

1)) we will prove that each side is subset of other

We will prove that each side is subset of other

Let $X \in (A \cap B)$

$\Rightarrow X \notin (A \cap B)$

$\Rightarrow \neg\ (X \in A \cap B)$

$\Rightarrow \neg\ (X \in A \wedge X \in B)$

$\Rightarrow X \notin A \vee X \notin B$

$\Rightarrow X \in \overline{A} \vee X \in \overline{B}$

$\Rightarrow X \in \overline{A} \cup \overline{B}$

$\Rightarrow (\overline{\overline{A} \cap \overline{B}}) \subseteq \overline{A} \cup \overline{B}$

Let $X \in \overline{A} \cup \overline{B}$

$\Rightarrow X \in \overline{A} \vee X \in \overline{B}$

$\Rightarrow X \notin A \vee X \notin B$

$\Rightarrow \neg\ (X \in A) \vee \neg\ (X \in B)$

$\Rightarrow \neg\ (X \in A \wedge X \in B)$

$\Rightarrow \neg\ (X \in A \cap B)$

$\Rightarrow X \notin (A \cap B)$

$\Rightarrow X \in (\overline{A \cap B})$

$\Rightarrow \overline{A} \cup \overline{B} \subseteq (\overline{A \cap B})$

2)) Use set builder notation and logical equivalences to establish the second De Morgan law (A ∩ B) = A ∪ B

$\overline{A} \cap \overline{B} = \{x / x \in \overline{A} \cup \overline{B}\}$

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Note that besides the definitions of complement, union, set membership, and set builder notation, this proof uses the first De Morgan law for logical equivalences.

3)) Use membership table to prove A∩(B∪C)=(A∩B)∪(A∩C)

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

EXAMPLES:

EX1// By laws

Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

EX2// Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by both side

Solution:

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A \wedge x \in (B \cup C)$

$\Rightarrow x \in A \wedge (x \in B \vee x \in C)$

$\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$

$\Rightarrow x \in A \cap B \vee x \in A \cap C$

$\Rightarrow x \in A \cap B \cup A \cap C$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Let $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in A \cap B \vee x \in A \cap C$

$\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$

$\Rightarrow x \in A \wedge (x \in B \vee x \in C)$

$\Rightarrow x \in A \wedge x \in (B \cup C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

H.W//

1. Let A and B two sets, Prove in one side :
   $A-B= A \cap \bar{B}$

2. Let A and B two sets, and $A \subseteq B$, show that $\bar{B} \subseteq \bar{A}$

3. Let A,B,C three sets and $A \subseteq B$ and $B \subseteq C$, show that $A \subseteq C$

EX3// Prove $(\bar{\bar{A}}) = A$

Solution:

Let $x \in (\bar{\bar{A}})$

$\Rightarrow x \notin \bar{A}$

$\Rightarrow x \in A$

$\Rightarrow (\bar{\bar{A}}) \subseteq A$

Let $x \in A$

$\Rightarrow x \notin \bar{A}$

$\Rightarrow x \in (\bar{\bar{A}})$

$\Rightarrow A \subseteq (\bar{\bar{A}})$

- **Computer Representation of Sets**

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an **arbitrary ordering** of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance a1 , a2 , . . . , an . Represent a subset A of U with the bit string of length n , where the i th bit in this string is 1 if aj belongs to A and is 0 if aj does not belong to A .

EX: Let U = { l , 2, 3 , 4, 5 , 6, 7 , 8 , 9, 10} , and the ordering of elements of U has the elements in increasing order; that is, aj = i . What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U ?

Solution:

The bit string that represents the set of **odd** integers i n U, namely, { 1 , 3 , 5 , 7, 9}. It is 1 0 1 0 1 0 1 0 1 0.

The bit string that represents the set of all **even** integers in U, namely, {2, 4, 6, 8, 10} , by the string 0 1 0 1 0 1 0 1 0 1 .

The set of all integers in U that do not **exceed 5** , namely, { l , 2 , 3 , 4 , 5 } ,
is represented by the string 1 1 1 1 1 0 0 0 0 0 .

NOTE// To obtain the bit string for the union and intersection of 2 sets we perform bitwise Boolean operation on bit string

Union as OR

Intersection as AND

EX1// We have seen that the bit string for the set { 1 , 3, 5 , 7, 9} (with universal set { l , 2, 3, 4, 5 , 6, 7, 8 , 9, 10} is 10 1 0 1 0 1 0 1 0. What is the bit string for the complement of this set ?

Solution:

The bit string for the complement of this set is obtained by replacing Os with 1 s and vice versa. This yields the string 0 1 0 1 0 1 0 1 0 1 , which corresponds to the set {2, 4, 6, 8, 10} .

EX2// Let A1 **is** { 1 , 3, 5 , 7, 9}, A2 is { l , 2, 3, 4, 5} , Find bit string of $A1 \cap A2$, $A1 \cup A2$?

Solution:

A1= { 1 , 3, 5 , 7, 9}= 1 0 1 0 1 0 1 0 1 0 , A2= { l , 2 , 3 , 4, 5}= 1 1 1 1 1 0 0 0 0 0

$A1 \cap A2$= 1 0 1 0 1 0 0 0 0 0

$A1 \cup A2$= 1 1 1 1 1 0 1 0 1 0

1. **Multisets**

Multisets are unordered collections of elements where an element can occur as a member more than once.

The notation {m1. a1 , m2 . a2 , . . . , mr . ar} denotes the multiset with element a1 occurring m1 times, element a2 occurring m2 times, and so on.

The numbers m j, i = 1 , 2, . . . , r are called the multiplicities of the elements aj , i = 1 , 2 , . . . , r .

- Let P and Q b e multi sets.

The **union** of the multisets P and Q is the multi set where the multiplicity of an element is the **maximum** of its multiplicities in P and Q .

The **intersection** of P and Q is the multi set where the multiplicity of an element is the **minimum** of its multiplicities in P and Q .

The **difference** of P and Q is the multi set where the multiplicity of an element is the **multiplicity** of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0.

The **sum** of P and Q is the multiset where the multiplicity of an element is the **sum of multiplicities** in P and Q .

NOTE// The union, intersection, and difference of P and Q are denoted by P U Q , P ∩ Q, and P - Q, respectively. The sum of P and Q is denoted by P + Q.
EX// let A,B two multisits:
A={ 3.a,2.b,1.c} , B ={ 2.a, 3.b, 4.d},
find A∪B, A∩B, A+B, A-B, B-A?

Solution:
A∪B= {3.a,3.b,1.c,4.d}
A∩B= {2.a,2.b}
A+B={5.a,5.b,1.c,4.d}
A-B={1.a,1.c}
B-A={1.b, 4.d}

2. Fuzzy sets

Fuzzy sets are used in artificial intelligence. Each element in the set has a degree of membership, which is a real number between 0 and 1 (including 0 and 1).

The **union** of two fuzzy sets S and T is the fuzzy set S ∪T, where the degree of membership of an element in S ∪T is **the maximum** of the degrees of membership of this element in S and in T.

The **intersection** of two fuzzy sets S and T is the fuzzy set S ∩ T, where the degree of membership of an element in S ∩ T is **the minimum** of the degrees of membership of this element in S and in T .

EX// let A={0.4a,0.8b, 0.3c} , B={0.1a,0.9b,0.2d}, find A∪B, A∩B?

Solution:
A∪B={0.4a,0.9b,0.3c,0.2d}
A∩B={0.1a,0.8b}

3. Cartesian product

Let A and B be sets. The Cartesian product of A and B, denoted by A x B, is the set of all ordered pairs (a , b), where a $\in$ A and b $\in$ B . Hence,
A x B = {(a , b)| a $\in$ A $\wedge$ b $\in$ B } .

EX// What is the Cartesian product of A = { 1 , 2} and B = {a , b, c}?
Solution:
The Cartesian product A x B is
A x B = { (1 , a), (1 , b) , (1 , c), (2, a), (2, b) , (2, c)} .
A subset R of the Cartesian product A x B i s called a relation from the set A to the set B . The elements of R are ordered pairs, where the first element belongs to A and the second to B .
For example, R = {(a , 0), (a , 1), (a , 3), (b, 1), (b, 2), (c, 0), (c, 3)} is a relation from the set {a , b , c} to the set {0, 1 , 2, 3 } . The Cartesian products A x B and B x A are not equal, unless A = 0 or B = 0 (so that A x B = 0) or A = B

Definition// The **Cartesian product** of the n sets A1 , A2 , . . . , An , denoted by A1 x A2 x • • • x An , is the set of ordered n -tuples (a 1 , a2 , . . . , an), where aj belongs to Aj for i = 1 , 2 , . . . , n . In other words, A1 X A2 x · · · x An = {(a1 , a2 , . . . , an)| aj $\in$ Ai for i = 1 , 2, . . . , n }.

EX// What is the Cartesian product A x B x C , where A = {0, l }, B = { 1 , 2}, and C = {0, 1 , 2}?

Solution:
The Cartesian product A x B x C consists of all ordered triples (a , b, c),
where a $\in$ A, b $\in$ B, and c $\in$ C. Hence,
A x B x C = {(0 , 1 , 0), (0, 1 , 1), (0, 1 , 2), (0, 2 , 0), (0, 2 , 1) , (0, 2, 2), (1 , 1 , 0), (1 , 1 , 1),(1 , 1 , 2), (1 , 2, 0), (1 , 2, 1), (1 , 2, 2)} .

- The Cartesian product is distributive on union and intersection

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$\rightarrow$ LET $(x,y) \in A \times (B \cup C)$

$x \in A \wedge y \in (B \cup C)$

$x \in A \wedge (y \in B \vee y \in C)$

$x \in A \wedge y \in B \vee x \in A \wedge y \in C$

$(x,y) \in (A \times B) \vee (x,y) \in (A \times C)$

$(x,y) \in (A \times B) \cup (A \times C)$

$A \times (A \cup C) \subseteq (A \times B) \cup (A \times C)$

$\rightarrow$ LET $(x,y) \in (A \times B) \cup (A \times C)$

$(x,y) \in (A \times B) \vee (x,y) \in (A \times C)$

$x \in A \wedge y \in B \vee x \in A \wedge y \in C$

$x \in A \wedge (y \in B \vee y \in C)$

$x \in A \wedge y \in (B \cup C)$

$(x,y) \in A \times (B \cup C)$

$(A \times B) \cup (A \times C) \subseteq A \times (A \cup C)$

$\therefore A \times (A \cup C) = (A \times B) \cup (A \times C)$

EX// Prove $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

$\rightarrow$LET $(x,y) \in (A \times B) \cap (C \times D)$

$(x,y) \in (A \times B) \wedge (x,y) \in (C \times D)$

$x \in A \wedge y \in B \wedge x \in C \wedge y \in D$

$x \in A \cap C \wedge y \in B \cap D$

$\therefore (x,y) \in (A \cap C) \times (B \cap D)$

$\rightarrow$LET $\therefore (x,y) \in (A \cap C) \times (B \cap D)$

$x \in (A \cap C) \wedge y \in (B \cap D)$

$x \in A \wedge x \in C \wedge y \in B \wedge y \in D$

$(x,y) \in (A \times B) \wedge (x,y) \in (C \times D)$

$\therefore (x,y) \in (A \times B) \cap (C \times D)$

CHAPTER TWO

- **Sequences**
- **Properties of Integers**

• Sequences

A sequence is a discrete structure used to represent an ordered list. For example, 1 , 2, 3, 5, 8 is a sequence with five terms and 1 , 3, 9, 27, 81 , . . . , 130, . . . is an infinite sequence.

A sequence is a function from a subset of the set of integers (usually either the set {0, 1, 2, . . . } or the set { 1 , 2 , 3, . . . J) to a set S. We use the notation an to denote the image of the integer n . We call an a term of the sequence.

EX// Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1 , namely, starts with n=

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Definition// A **geometric progression** is a sequence of the form a , ar, ar^2 ,…, ar^n , … where the **initial term a** and the **common ratio r** are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^X$.

EX//

The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and −1; 2 and 5; and 6 and 1/3, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with

$$1, -1, 1, -1, 1, \ldots;$$

The list of terms Co , C1 , C2 , C3 , C4 , • • • begins with 2, 10 , 50, 250, 1250, . . . ; and the list of terms do , d1, d2 , d3 , d4 , • • . begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$

Definition// An **arithmetic progression** is a sequence of the form a , a + d, a + 2d,…, a + nd, . . .
where the **initial term a** and the **common difference d** are real numbers.
Remark: An arithmetic progression is a discrete analogue of the linear function f(x) = a+ dx .

EX// The sequences {sn} with Sn = - 1 + 4n and {Tn } with fn = 7 - 3n are both arithmetic progressions with initial terms and common differences equal to - 1 and 4, and 7 and -3, respectively, if we start at n = 0.
The list of terms So , Sl , S2 , S3 , . . . begins with - 1 , 3 , 7, 11 ,…, and the list of terms f0 , f1 , f2 , f3 , . . . begins with 7, 4, 1 , -2, ...

The **finite sequences** are also called **strings**.
The **length** of the string S is the number of terms in this string.
The **empty** string, is the string that has no terms. The empty string has length zero.

- ## Special Integer Sequences

A common problem in discrete mathematics is finding a formula or a general rule for constructing the terms of a sequence.
There are many questions you could ask, but some of the more useful are:

1. Are there runs of the same value? That is, does the same value occur many times in a row?
2. Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
3. Are terms obtained from previous terms by multiplying by a particular amount?
4. Are terms obtained by combining previous terms in a certain way?
5. Are there cycles among the terms?

EX// Find formulae for the sequences with the following first five terms:
(a) 1 , 1/2, 1 /4, 1 /8, 1 / 16
(b) 1 , 3 , 5, 7, 9

Solution:

(a) The sequence with $a_n = (1/2)^n$, n = 0, 1 , 2, . . . is a possible match. This proposed sequence is a **geometric progression** with a = 1 and r = 1 /2.

(b) The sequence with $a_n = 1+2n$, n = 0, 1 , 2, . . . is a possible match. This proposed sequence is an **arithmetic progression** with a = 1 and d = 2.

EX// How can we produce the terms of a sequence if the first 10 terms are 5, 11 , 17, 23, 29, 35, 41 , 47, 53 , 59?

Solution:

Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutive terms is 6) Consequently, the nth term could be produced by starting with 5 and adding 6 a total of n - 1 times; that is, a reasonable guess is that the nth term is 5 + 6(n - 1) = 6n - 1 . (This i s an arithmetic progression with a = 5 and d = 6.)

In table1 some useful sequences:

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .

- **Summations**

Summations from the sequence $a_m, a_{m+1}, \ldots, a_n$, We use the notation

$$\sum_{j=m}^{n} a_j , \qquad \sum_{j=m}^{n} a_j , \qquad \text{or} \qquad \sum_{1 \le j \le n} a_j$$

Here, the variable j is called the index of summation, it runs throw the lower limit m and the upper limit n.

EX// Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1 / n$ for $n = 1 , 2, 3 , \ldots .$
Solution: The lower limit for the index of summation is 1 , and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$

What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\begin{aligned} \sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55. \end{aligned}$$

What is the value of $\sum_{k=4}^{8} (-1)^k$?

Solution: We have

$$\begin{aligned} \sum_{k=4}^{8} (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1. \end{aligned}$$

Double summations arise in many contexts (as in the analysis of nested loops in computer programs).

An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$
$$= \sum_{i=1}^{4} 6i$$
$$= 6 + 12 + 18 + 24 = 60.$$

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set.
That is, we write

$$\sum_{s \in S} f(s)$$

What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution:

$= 0 + 2 + 4 = 6.$

TABLE 2: Some Useful summation Formulae

$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$

EX 1//

Find $\sum_{k=50}^{100} k^2$.

Solution: First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2, we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

H.W1// What are the values of these sums, where S = {1, 3 , 5 , 7}?

a) $\sum_{j \in S} j$ **b)** $\sum_{j \in S} j^2$

c) $\sum_{j \in S} (1/j)$ **d)** $\sum_{j \in S} 1$

H.W2// What is the value of each of these sums of terms of a geometric progression?

a) $\sum_{j=0}^{8} 3 \cdot 2^j$ **b)** $\sum_{j=1}^{8} 2^j$

c) $\sum_{j=2}^{8} (-3)^j$ **d)** $\sum_{j=0}^{8} 2 \cdot (-3)^j$

H.W3// Compute each of these double sums.

a) $\sum_{i=1}^{2} \sum_{j=1}^{3} (i+j)$ **b)** $\sum_{i=0}^{2} \sum_{j=0}^{3} (2i+3j)$

c) $\sum_{i=1}^{3} \sum_{j=0}^{2} i$ **d)** $\sum_{i=0}^{2} \sum_{j=1}^{3} ij$

➢ The Properties of Integers

The part of mathematics involving the integers and their properties belongs to the branch of mathematics called **number theory**.

1. Division

Definition// If a and b are integers with a # 0, we say that a divides b if there is an integer c such that b = ac. When a divides b we say that a is a factor of b and that b is a multiple of a . The notation a | b denotes that a divides b. We write a |/ b when a does not divide b.

EX//Determine whether 3 |7 and whether 3 |12.
Solution:
It follows that 3 |/7, because 7/3 is not an integer. On the other hand, 3 |12 because 12/3 = 4.

Definition// In the equality given in the division algorithm, d is called the divisor, a is called the dividend, q is called the quotient, and r is called the remainder. this notation is used to express the quotient and remainder:
q = a div d, r = a mod d.

EX// What are the quotient and remainder when 101 is divided by 11?
Solution: We have 101= 99 + 2.
Hence, the quotient when 101 is divided by 11 is 9 = 101 div 11 , and the remainder is 2 = 101 mod 11 .

2. Primes

Every positive integer greater than 1 is divisible by at least two integers, because a positive integer is divisible by 1 and by itself. Positive integers that have exactly two different positive integer factors are called primes.

- A positive integer that is greater than 1 and is not prime is called **composite**.

Remark: The integer n is composite if and only if there exists an integer a such that a|n and 1 < a < n .

EX// The integer 7 is prime because its only positive factors are 1 and 7, whereas the integer 9 is composite because it is divisible by 3 .

The primes less than 100 are 2, 3 , 5, 7, 11 , 13 , 17, 19, 23, 29, 31 , 37, 41 , 43, 47, 53, 59, 61 ,67, 71 , 73, 79, 83, 89, and 97.

EX// The prime factorizations of 100, 999, and 1024 are given by

$100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$,
$999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$,
$1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$.

EX// Find the prime factorization of 7007.

Solution:
To find the prime factorization of 7007, first perform divisions of 7007 by successive primes, beginning with 2. None of the primes 2, 3 , and 5 divides 7007.
However, 7 divides 7007, with 7007/7 = 1001 . Next, divide 1001 by successive primes, beginning with 7. It is immediately seen that 7 also divides 1001 , because 1001 /7 = 143 .
Continue by dividing 143 by successive primes, beginning with 7.
Although 7 does not divide 143, 11 does divide 143, and 143 / 11 = 13 .
Because 13 is prime, the procedure is completed. It follows that the prime factorization of 7007 is $7 \cdot 7 \cdot 11 \cdot 13 = 7^2 \cdot 11 \cdot 13$.

3. Greatest Common Divisors (GCD)

The largest integer that divides both of two integers is called the greatest common divisor of these integers.
Let a and b be integers, not both zero. The largest integer d such that d | a and d | b is called the greatest common divisor of a and b. The greatest common divisor of a and b is denoted by gcd (a , b).

EX// What is the greatest common divisor of 24 and 36?
Solution: The positive common divisors of 24 and 36 are 1 , 2, 3 , 4, 6, and 12 . Hence, gcd(24, 36) = 12 .

EX// What is the greatest common divisor of 17 and 22?
Solution: The integers 17 and 22 have no positive common divisors other than 1 , so that gcd(17, 22) = 1 .

- **The second way** to find the greatest common divisor of two integers is to use the **prime factorizations** of these integers. Suppose that the prime factorizations of the integers a and b, neither equal to zero, are

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}, \; b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n},$$

where each exponent is a nonnegative integer, and where all primes occurring in the prime factorization of either a or b are included in both factorizations, with zero exponents if necessary. Then gcd(a , b) is given by

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_n, b_n)},$$

where min(x , y) represents the minimum of the two numbers x and y. To show that this formula for gcd(a , b) is valid, we must show that the integer on the right-hand side divides both a and b, and that no larger integer also does. This integer does divide both a and b, because the power of each prime in the factorization does not exceed the power of this prime in either the factorization of a or that of b. Further, no larger integer can divide both a and b, because the exponents of the primes in this factorization cannot be increased, and no other primes can be included.

EX/ Because the prime factorizations of 120 and 500 are $120 = 2^3 \cdot 3 \cdot 5$ and $500 = 2^2 \cdot 5^3$, the greatest common divisor is

$$\gcd(120, 500) = 2^{\min(3, 2)} 3^{\min(1, 0)} 5^{\min(1, 3)} = 2^2 3^0 5^1 = 20.$$

EX// Find GCD of each by finding the prime factorizations?

1. gcd (2415,3289)
   2415=3.5.7.23
   3289=11.13.23
   gcd (2415,3289) =23

2. gcd (406,555)
   406=2.7.29
   555=3.5.37
   gcd (406,555) =1

3. gcd (4278,8602)
   4278=2.3.23.31
   8602=2.11.17.23
   gcd (4278,8602) =2.23=46

- **The third way to find GCD is Euclidean Algorithm**

The Euclidean Algorithm

This method asks you to perform successive division:

1. The smaller of 2 number into the larger
2. The resulting remainder divided into divisor until the remainder is equal zero, of that point look to the remainder of the previous division that will be the greatest common divisor.

The Euclidean Algorithm.

```
procedure gcd(a , b: positive integers)
{x := a
y := b
while (y =! 0)
begin
r := x mod y
x := y
y := r
end (gcd(a , b) is x }
```

In Algorithm , the initial values o f x and y are a and b, respectively. At each stage of the procedure, x is replaced by y, and y is replaced by x mod y, which is the remainder when x is divided by y. This process is repeated as long as $y \neq 0$. The algorithm terminates when y = 0, and the value of x at that point, the last nonzero remainder in the procedure, is the greatest common divisor of a and b.

EX// Find the greatest common divisor of 414 and 662 using the Euclidean algorithm.

Solution:
Successive uses of the division algorithm give:
x =662, y =414, r =248
x =414, y =248, r =166
x =248, y =166, r =82
x =166, y =82 , r =2
x =82, y =2, r =0
x =2, y =0

Hence, gcd(414, 662) = 2, because 2 is the last nonzero remainder.

EX// Find the gcd (1424,3084) using Euclidean

Solution:
Successive uses of the division algorithm give:
x =3084, y =1424, r =236
x =1424, y =236, r =8
x =236, y =8, r =4
x =8, y =4, r =0
x =4, y =0

Hence, gcd(1424,3084) = 4, because 4 is the last nonzero remainder.

Applications of Number Theory

Congruence's have many applications to discrete mathematics and computer science. One of these is cryptology, which is the study of secret messages. One of the earliest is Julius Caesar by shifting each letter three letters forward in the alphabet.
In the encrypted version of the message, the letter represented by p is replaced with the letter represented by (p + 3) mod 26.
0 1 2 3 4 5 6 7 8 9…….25
a b c d e f g h i j…….z

The P integer p<=25 can be replaced by
£(p)=(p+3)mod 26
Let p= 6 which is the char 9
£(p)= (6+3) mod 26
=9 which is the char j
Then we replace g by j

EX// What is the secret message produced from the message "MEET YOU IN THE PARK" using the Caesar cipher?
Solution: First replace the letters in the message with numbers.

This produces
12 4 4 19 24 14 20 8 13 19 7 4 15 0 17 10.

Now replace each of these numbers p by
f(p) = (p + 3) mod 26.
This gives
I5 7 7 22 1 17 23 11 16 22 10 7 18 3 20 13 .

Translating this back to letters produces the encrypted message
"PHHW B RX LQ WKH SDUN."

The decryption
£′(p)= (p-3) mod 26
One approach to enhance the security is to use a function of the form
£(p)=(ap+b) mod 26

Q:

1. What is the secret message produced from the message "EOXH MHDQV" using the Caesar cipher?
2. What is the secret message produced from the message "HDW GLP VXP" using the Caesar cipher?

Hashing Functions

The central computer at an insurance company maintains records for each of its customers. How can memory locations be assigned so that customer records can be retrieved quickly? The solution to this problem is to use a suitably chosen hashing function.

Records are identified using a key, which uniquely identifies each customer's records. For instance, customer records are often identified using the Social Security number of the customer as the key. A hashing function h assigns memory location h (k) to the record that has k as its key.

In practice, many different hashing functions are used. One of the most common is the Function h(k) = k mod m
where m is the number of available memory locations.

Hashing functions should be easily evaluated so that files can be quickly located. The hashing function h (k) = k mod m meets this requirement; to find h (k), we need only compute the remainder when k is divided by m . Furthermore, the hashing function should be onto, so that all memory locations are possible. The function h (k) = k mod m also satisfies this property.

EX1// Let m=10, find hashing function to 95,90,45

Solution:

1. H(95)= 95 mod 10= 5
2. H(90)= 90 mod 10= 0
3. H(45)= 45 mod 10= 5

To solve this assign the first free location, the following location 45 set to 6 .

90	0
	1
.	
.	
.	
95	5
45	6
.	
.	
	9

EX2// parking has (31) visitor spaces, numbered from 0 to 30. Visitation use hashing function h(k)= k mod 31, where k is the number of the car : 317, 918, 100, 111, 310.

Solution:

1. h(317)= 317 mod 31= 7
2. h(917)= 917 mod 31= 18
3. h(100)= 100 mod 31= 7
4. h(111)= 111 mod 31= 18
5. h(310)= 310 mod 31= 0

310	0
	1
.	
.	
317	7
100	8
.	
.	
918	18
111	19
.	
.	
	30

The RSA Cryptosystem

1. It is a public key cryptosystem.
2. It is based on modular exponentiation modulo the product of two large primes.
3. Each individual has an encryption key consisting of a modulus n = pq , where p and q are large primes, and **m** =(p - 1)(q - 1).
4. Choose a small number e such that gcd(e,m)=1
5. Find **d**, such that d= (1+zm)/e , where z any integer.

RSA Encryption

In the RSA encryption method, messages are translated into sequences of integers. This can be done by translating each letter into an integer, as is done with the **Caesar cipher**. These integers are grouped together to form larger integers, each representing a block of letters. The encryption proceeds by transforming the integer M , representing the plaintext (the original message), to an integer C , representing the ciphertext (the encrypted message), using the function

$C = M^e \mod n$ using public key(n,e).

(To perform the encryption, we use an algorithm for fast modular exponentiation, We leave the encrypted message as blocks of numbers and send these to the intended recipient)

RSA Decryption

The plaintext message can be quickly recovered when the decryption key d, an inverse of e modulo (p-1)(q-1), is known. [Such an inverse exists because gcd(e,(p-1)(q-1))= 1 To see this, note that if de == 1 (mod (p - 1) (q -1)), there is an integer k such that de = 1 + k(p - 1)(q – 1). It follows that

$C^d \equiv (M^e)^d = M^{de} = M^{1+k(p-1)(q-1)} (\text{mod } n)$. using secret key(n,d)

By Fermat's Little Theorem [assuming that gcd(M , p) = gcd(M, q) = 1 , which holds except in rare cases], it follows that MP- l == 1 (mod p) and Mq- 1 == 1 (mod q). Consequently

C^d == M . $(M^{P-1})^{K(q-l)}$ == M . 1 == M (mod p)

And

C^d == M · $(M^{q-1})^{k(p-1)}$ == M · 1 == M (mod q)

Because gcd(p , q) = 1, it follows by the Chinese Remainder Theorem that

C^d == M (mod pq).

EX1// Encrypt the message STOP with p=11, q=13?

Solution:

N=p.q=11.13=143

M= (p-1)(q-1)= 10.12=120

Find e such that Gcd(e,m)=1

E=2,3,4,5,6,7

Gcd(e,120)=1

∴E=7

We compute $c=m^e$ mod n S T O P

$C=(18\ 19)^5$ mod 143=k1 18 19 14 15

$C=(14\ 15)^5$ mod 143=k2

K1 k2 is crypt message

To decrypt the message :

We compute d= (1+zm)/e , where z any integer , d integer

Z=0, d =1/7

Z=1, d =(1+120) /7

Z=6, d =(1+6*120)/7= 103

$M=c^d$ mod n.

EX2// Encrypt the message BASRAH with p=7, q=11

Solution:

N=p.q=7.11=77
M= (p-1)(q-1)= 6.10=60

Find e such that Gcd(e,m)=1
E=2,3,4,5,6,7
Gcd(e,60)=1
$\therefore$ E=7

We compute $c=m^e \bmod n$ B A S R A H
$C=(10)^7 \bmod 77=k1$ 1 0 18 17 0 7
$C=(18\ 17)^7 \bmod 77=k2$
$C=(0\ 7)^7 \bmod 77=k3$
K1 k2 K3 is crypt message

To decrypt the message :
We compute d= (1+zm)/e , where z any integer , d integer
d=(1+60*z)/7
z==0,1,2,3,4,5
$\therefore$ d=(1+60*5)/7 = 301/7=43
$M=c^d \bmod n$.
$M1=(k1)^{43} \bmod n$, $M2=(k2)^{43} \bmod n$, $M3=(k3)^{43} \bmod n$

CHAPTER THREE

- Matrices
- Propositional and Logical Operations
- Conditional Statements

- **Matrices**

Matrices are used throughout discrete mathematics to express relationships between elements in sets. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models.

Definition// A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an m x n matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called **square**. Two matrices are **equal** if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

EXAMPLE 1

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix.

We now introduce some terminology about matrices. Boldface uppercase letters will be used to represent matrices.

Definition// Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The ith *row* of $\mathbf{A}$ is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The jth *column* of $\mathbf{A}$ is the $n \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{bmatrix}.$$

The (i, j)th element or entry of A is the element aij, that is, the number in the ith row and jth column of A. A convenient shorthand notation for expressing the matrix A is to write A = [aij], which indicates that A is the matrix with its (i, j)th element equal to aij.

- **Matrix Arithmetic**
  The basic operations of matrix arithmetic will now be discussed.

  - **The sum of two matrix**

Definition// Let A = [aij] and B = [bij] be m x n matrices. The sum of A and B, denoted by A + B, is the m x n matrix that has aij + bij as its (i, j)th element. In other words, A + B = [aij + bij].
The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, because the sum of two matrices is defined only when both matrices have the same number of rows and the same number of columns.

EX//

$$\text{We have } \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}.$$

  - **The product of two matrix**

Definition// Let A be an m x k matrix and B be an k x n matrix. The product of A and B, denoted by AB, is the m x n matrix with its (i , J)th entry equal to the sum of the products of the corresponding elements from the ith row of A and the Jth column of B. In other words,

if AB = $[c_{ii}]$ then $c_{ij} = a_{i1}\, b_{1j} + a_{i2}b_{2i} + \ldots + a_{ik}b_{ki}$.

A product of two matrices is defined only when the number of columns in the first matrix equals the number of rows of the second matrix.

EX// Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find AB if it is defined.

Solution: Because A is a 4 x 3 matrix and B is a 3 x 2 matrix, the product AB is defined and is a 4 x 2 matrix.
To find the elements of AB, the corresponding elements of the rows of A and the columns of B are first multiplied and then these products are added.

AB are computed, we see that

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}.$$

- **The commutative product of two matrix**

Matrix multiplication is not commutative. That is, if A and B are two matrices, it is not necessarily true that AB and BA are the same. In fact, it may be that only one of these two products is defined. For instance, if A is 2 x 3 and B is 3 x 4, then AB is defined and is 2 x 4; however, BA is not defined, because it is impossible to multiply a 3 x 4 matrix and a 2 x 3 matrix.

Definition// Let A is an m x n matrix and B is an r x s matrix. Then AB is defined only when n = r and BA is defined only when s = m. Moreover, even when AB and BA are both defined, they will not be the same size unless m = n = r = s .
Hence, if both AB and BA are defined and are the same size, then both A and B must be square and of the same size. Furthermore, even with A and B both n x n matrices, AB and BA are not necessarily equal.

EX// Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

Solution: We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$.

- **Algorithms for Matrix Multiplication**

The definition of the product of two matrices leads to an algorithm that computes the product of two matrices. Suppose that C = [cij] is the m x n matrix that is the product of the m x k matrix A = [aij] and the k x n matrix B = [bij]. The algorithm based on the definition of the matrix product is expressed in pseudocode in Algorithm .

Algorithm Matrix Multiplication.

Procedure matrix multiplication(A, B: matrices)

for i := 1 to m
for j := 1 to n
begin
Cij := 0
for q := 1 to k
Cij := Cij + aiq bqj
end
{C = [Cij] is the product of A and B}

We can determine the complexity of this algorithm in terms of the number of additions and multiplications used.

EX// In which order should the matrices A1, A2 , and A3 where A1 is 30 x 20, A2 is 20 x 40, and A3 is 40 x 10, all with integer entries-be multiplied to use the least number of multiplications of integers?

Solution:

There are **two** possible ways to compute A1 (A2A3).

These are A1 (A2A3) and (A1 A2)A3 .

1. If A2 and A3 are first multiplied, a total of 20.40.10 = 8000 multiplications of integers are used to obtain the 20 x 10 matrix A2A3 . Then, to multiply A1 and A2A3 requires 30.20.10 = 6000 multiplications. Hence, atotal of 8000 + 6000 = 14,000 multiplications are used.

2. On the other hand, if A1 and A2 are first multiplied, then 30 .20 .40 =24,000 multiplications are used to obtain the 30 x 40 matrix A1 A2 . Then, to multiply A1 A2 and A3 requires 30 x 40 x 10 = 12,000 multiplications. Hence, a total of 24,000 + 12,000 = 36,000 multiplications are used. Clearly, the first method is more efficient.

- **Identity Matrix**

Definition// The **identity** matrix of order n is the n x n matrix

In = [Sij], where Sij = 1 if i = j and Sij = 0 if i # j .

Hence

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when A is an m x n matrix, we have

$$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

- **Powers of Matrices**

Powers of the matrices can be defined. When A is an n x n matrix, we have

$$\mathbf{A}^0 = \mathbf{I}_n, \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{r \text{ times}}.$$

- **Transposes of Matrices**

Definition// Let A = [aij] be an m x n matrix. The transpose of A, denoted by N, is the n x m matrix obtained by interchanging the rows and columns of A. In other words, if At = [hij], then hij = aji for i = 1 , 2 , ... , n and j = 1 , 2, ... ,m.

EX// The transpose of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is the matrix } \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Matrices that do not change when their rows and columns are interchanged are often important.

- **Symmetric Matrices**

Definition// A square matrix A is called symmetric if A = Á. Thus A = [aij] is symmetric if aij = aji for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.

Note that a matrix is symmetric if and only if it is square and it is symmetric with respect to its main diagonal (which consists of entries that are in the ith row and ith column for some i).

EX//

$$\text{The matrix } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is symmetric.}$$

- **Zero-One Matrices**

A matrix with entries that are either 0 or 1 is called a zero-one matrix. Algorithms using these structures are based on Boolean arithmetic with zero-one matrices. This arithmetic is based on the Boolean operations $\vee$ and $\wedge$, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

- **Join and Meet of the zero-one matrixes**

Definition// Let A = [aij] and B = [bij] be m x n zero-one matrices. Then the **join** of A and B is the zero-one matrix with (i, j)th entry aij $\vee$ bij. The join of A and B is denoted by A $\vee$ B. The **meet** of A and B is the zero-one matrix with (i, j)th entry aij $\wedge$ bij. The meet of A and B is denoted by A $\wedge$B.

EX// Find the **join** and **meet** of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: **We find that the join of A and B is**

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of A and B is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- **Boolean product of the zero-one matrixes**

Definition// Let A = [aij] be an m x k zero-one matrix and B = [bij] be an k x n zero-one matrix. Then the **Boolean product** of A and B, denoted by AOB, is the mxn matrix with (i, j)th entry Cij where Cij = (ail ∧ b1j) ∨ (ai2 ∧ b2j) ∨ ... ∨ (aik ∧ bkj).
Note that the Boolean product of A and B is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation ∨ and with multiplication replaced with the operation ∧. We give an example of the Boolean products of matrices.

Algorithm the boolean product.

procedure Boolean product (A, B: zero-one matrices)
{for i := 1 to m
for j := 1 to n
begin
Cij := 0
forq := 1 to k
Cij := Cij ∨ (aiq ∧ bqj)
End }
C = [cij] is the Boolean product of A and B

EX// Find the Boolean product of A and B, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

- **Boolean Power of the zero-one matrixes**

Definition// Let A be a square zero-one matrix and let r be a positive integer. The rth Boolean power of A is the Boolean product of r factors of A. The rth Boolean product of A is denoted by $A^{[r]}$.

Hence

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

EXAMPLE Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$. ◀

The number of bit operations used to find the Boolean product of two $n \times n$ matrices can be easily determined.

- **Propositional and Logical Operations**

1. Propositional Logic

A proposition is a declarative sentence that is either true or false, but not both.

EX// Consider the following sentences.

1 . What time is it?
2 . Read this carefully.
3 . x + 1 = 2 .
4 . x + y = Z.
5. Baghdad is the capital of Iraq.
6. 10+20=40.

Solution:

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables. Sentence 5 is true proposition but Sentence 6 is false proposition.

- Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

Definition// Let p be a proposition. The negation of p, denoted by ¬p (also denoted by ¯p), is the statement "It is not the case that p."

The proposition ¬ p is read "not p." The truth value of the negation of p, ¬ p, is the opposite of the truth value of p.

TABLE 1 The Truth Table for the Negation of a Proposition.	
P	¬p
T	F

EX//

"Today is Friday." → P

"Today is NOT Friday." → ¬P

Definition // Let p and q be propositions. The **conjunction** of p and q , denoted by $p \wedge q$, is the proposition "p and q ". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Table 2 displays the truth table for $p \wedge q$.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Note that in logic the word "**but**" sometimes is used instead of "**and**" in a conjunction. For example, the statement "The sun is shining, **but** it is raining" is another way of saying "The sun is shining **and** it is raining."

EX// Find the conjunction of the propositions p and q where p is the proposition "Today is Friday" and q is the proposition "It is raining today."

Solution: The conjunction of these propositions, $p \wedge q$, is the proposition "Today is Friday and it is raining today." This proposition is true on rainy Fridays and is false on any day that is not a Friday and on Fridays when it does not rain.

Definition// Let p and q be propositions. The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q ". The disjunction $(p \vee q)$ is **false** when both p and q are false and is **true** otherwise.

Table 3 displays the truth table for $p \vee q$.

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective or in a disjunction corresponds to one of the two ways the word **or** is used in English, namely, in an inclusive way. A disjunction is true when at least one of the two propositions is true. For instance, the inclusive or is being used in the statement "Students who have taken calculus or computer science can take this class."

Definition//Let p and q be propositions. The exclusive or of p and q,denoted by p⊕q, is the proposition that is **true** when exactly one of p and q is true and is **false** otherwise.
The truth table for the exclusive or of two propositions is displayed in Table 4.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

2. Conditional Statements

We will discuss several other important ways in which propositions can be combined.

Definition// Let p and q be propositions. The conditional statement p →q is the proposition "if p, then q ". The conditional statement p→ q is false when p is true and q is false, and true otherwise.
The truth table for the conditional statement p→ q is shown in Table 5.

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In the conditional statement p → q , p is called the **hypothesis** (or antecedent or premise) and q is called the **conclusion** (or consequence).
The statement p →q is called a **conditional statement** because p →q asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

EXAMPLE //
"If it is raining, then the home team wins." $p \rightarrow q$
The **contrapositive** ($\neg q \rightarrow \neg p$) is of this conditional statement is "If the home team does not win, then it is not raining."
The **converse** ($q \rightarrow p$) is "If the home team wins, then it is raining."
The **inverse** ($\neg p \rightarrow \neg q$)is "If it is not raining, then the home team does not win."
Only the **contrapositive** is equivalent to the original statement

Definition// Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q ." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called **bi-implications**.

- The last way of expressing the bi-conditional statement $p \rightarrow q$ uses the abbreviation "iff" for "if and only if". Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

The truth table for $p \leftrightarrow q$ is shown in Table 6.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EX // Let p be the statement "You can take the flight" and let q be the statement "You buy a ticket." Then p → q is the statement "You can take the flight if and only if you buy a ticket."

EX//Construct the truth table of the compound proposition
$(p \vee \neg q) \rightarrow (p \wedge q)$.
The resulting truth table is shown in Table 7.

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

TABLE 8 Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Table 8 displays the precedence levels of the logical operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$

3. Propositional Equivalences

Definition// A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a **tautology**. A compound proposition that is always false is called a **contradiction.**
A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

- **Tautology** $(p \vee \neg p)$, **Contradiction** $(p \wedge \neg p)$

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Definition // The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.
EX//$p \rightarrow q \equiv \neg p \vee q$

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

4. proposition Algebra

let p,q and r propositions

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

EX// Show that ¬ (p ∨ (¬ p ∧ q)) and (¬ p ∧ ¬ q) are logically equivalent by developing a series of logical equivalences.

Solution:

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law}\\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law}\\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law}\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law}\\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F}\\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction}\\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}$$

EX// is (p∧q)∧¬(p∨q) tautology or contradiction

(p∧q)∧¬(p∨q) = (p∧q)∧ (¬ p∧¬ q)

= p∧q ∧ ¬ p∧¬ q

= (p∧¬ p)∧ (q ∧¬ q)

= F ∧ F

= F → contradiction

H.W//

1. is [p∧(p→q)] → q tautology or contradiction

2. (¬ q∧(p→q))→ ¬q tautology or contradiction

3. prove p∧q→ p∨q is tautology

4. prove (p∧q)∧¬(p∨q) is contradiction

5. prove ((p→ q)∧¬q)→¬ p is tautology

6.Use truth table to show the prove is tautology or contradiction?

A. (p→q) ↔ (¬ q→ ¬ p)
B. q→ (p∨¬p)
C. ¬ [(p∧¬ p) → q)

5. Quantifiers

Definition// The universal quantification of P (x) is the statement "P(x) for all values of x in the domain."

The notation $\forall x\ P(x)$ denotes the **universal** quantification of P(x). Here $\forall$ is called the **universal quantifier**. We read $\forall x\ P(x)$ as "for all x P(x)" or "for every x P(x)." An element for which P(x) is false is called **a counterexample** of x P (x).
The meaning of the universal quantifier is summarized in the first row of Table 1

Definition// The **existential** quantification of P(x) is the proposition
"There exists an element x in the domain such that P (x)."
We use the notation $\exists x\ P(x)$ for the existential quantification of P(x). Here $\exists$ is called the **existential quantifier**.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Examples

Every student in comp. dept his age <25 years
There is student in IS dept. his age > 25 years
Every student in the class, has studied calculus.

EX// Some student in the class has visited cairo, and every one in the class has visited either Baghdad or cairo
Solution:
P(x): visited cairo, Q(x): visited baghdad , $\exists x\ p(x) \wedge \forall x\ (p(x) \vee q(x))$

EX// The sum of two positive integers is positive ?
Solution:
$\forall X\ \forall y\ (x>0 \wedge y>0) \rightarrow x+y>0$

EX// Let $p(x) \equiv x>3$ what are the truth values of p(4) and p(2)
Solution:
P(4) true, P(2) false

EX// Let $q(x,y) \equiv x=y+3$, what are the truth value of the propositions q(1,2), q(3,0).
Solution:
q(1,2) false, q(3,0) true

6. Negating Quantified Expressions

The negation for $\exists x\, p(x)$ is $\forall x \neg p(x)$

The negation for $\forall x\, p(x)$ is $\exists x \neg p(x)$

EX// There is student in your class who has taken a course in calculus: $\exists\, x\ p(x)$

Every student in your class, has not taken a course in calculus: $\forall\ x \neg p(x)$

EX// Let p(x) is "x student spend more than 4 hours daily in studying "

Express each of the quantifiers in English

- $\exists X\ p(x)$
  There is a student who spends more than 4 hours daily in studying

- $\forall X\ p(x)$
  Every student who spends more than 4 hours daily in studying.

- $\exists X \neg p(x)$
  There is a student who does not spend more than 4 hours daily in studying.

- $\forall X \neg p(x)$
  Every student who does not spends more than 4 hours daily in studying.

EX// Let L (x , y) be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody loves Mohammad.
   $\forall\ x\ L$ (x, Mohammad)
b) Everybody loves somebody.
   $\forall\ x\ \exists\ y\ L$ (x,y)
c) There is somebody whom everybody loves him.
   $\exists y\ \forall x\ L$ (x,y)
d) Nobody loves everybody.
   $\forall x\ \forall y \neg\ L$ (x,y)
e) There is somebody whom ahmad does not love.
   $\exists y \neg L$(ahmad,y)

t) There is somebody whom no one loves.
   $\exists y\ \forall x \neg L(x,y)$

i) Everyone loves himself.
   $\forall x\ L$ (x,x).

CHAPTER FOUR

- **Mathematical Induction**
- **Recursive**
- **Methods of Proving Theorems**

➢ Mathematical Induction

In general, mathematical induction can be used to prove statements that assert that P(n) is true for all positive integers n, where P (n) is a propositional function. A proof by mathematical induction has two parts, a basis step, where we show that P(l) is true, and an inductive step, where we show that for all positive integers k, if P(k) is true, then P(k + 1) is true.

Principle of mathematical induction to prove that P(n) is true for all positive integers n , where P(n) is a propositional function, we complete two steps:

1. Basis step: we verify that p(k) is true.
2. Inductive step: we show that the conditional statement p(k) $\rightarrow$ p(k + 1) is true for all positive integers k.

Examples of Proofs by Mathematical Induction

EX1// use mathematical induction to prove for all n, 1+2+… +n= n(n+1)/2 ?

Solution:
The basic idea is to approach the proof in 2 step

1. Prove the statement is true for n=1 (basis step)
2. Prove that whenever the statement is true for case n, it is also true for n+1 (inductive step)

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers is $n(n+1)/2$
We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show tha
$P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$

BASIS STEP: $P(1)$ is true, because $1 = \dfrac{1(1+1)}{2}$.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrar
positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Means p(k+1) is ture

EX2// use mathematical induction to prove the sum the first n odd integer is n^2 ?

$1+3+5+\ldots..+(2n-1)=n^2$

Solution:

First step :p(1) is true $1=1^2$

Induction Step: assume p(k) is true so p(k+1) is true

$1+3+5+\ldots..+(2k-1)=k^2$

$p(k+1)= 1+3+5+\ldots..+2(k+1-1)=(k+1)^2$

$=K^2+2k+1= (k+1)^2$

We must prove $p(k)\rightarrow p(k+1)$

$1+3+5+\ldots..+(2k-1)+ (2(k+1)-1)=k^2+2k+1$

$=(k+1)^2$

EX3//Use mathematical Induction to show :

$1+2+2^2+\ldots+2^n=2^{n+1}-1$ for n=0,1,..

Solution:

Basis step :p(0) is true since $2^0=2-1=1$

Induction Step: assume p(k) is true

$1+2+2^2+\ldots+2^k=2^{k+1}-1$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1+2+2^2+\cdots+2^k+2^{k+1}=2^{(k+1)+1}-1=2^{k+2}-1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned}1+2+2^2+\cdots+2^k+2^{k+1}&=(1+2+2^2+\cdots+2^k)+2^{k+1}\\&=(2^{k+1}-1)+2^{k+1}\\&=2\cdot 2^{k+1}-1\\&=2^{k+2}-1.\end{aligned}$$

EX4//Use mathematical Induction to show :

$1.2+2.3+\ldots+n(n+1)= n(n+1)(n+2)/3$

Solution:

Basis step :p(1) is true since

$1*2=1(1+1)(1+2)/3$

$2=2$

Induction Step: assume p(k) is true

$1.2+2.3+\ldots+k(k+1)= k(k+1)(k+2)/3$

P(k+1)=

$1.2+2.3+\ldots+(k+1)(k+2) = (k+1)(k+2)(k+3)/3$

We must prove $p(k)\rightarrow p(k+1)$

$1.2+2.3+\ldots+k(k+1)+(k+1)(k+2) = k(k+1)(k+2)/3+(k+1)(k+2)$

$=(k+1)(k+2)\{k/3+1\}$

$=(k+1)(k+2)(k+3)/3$

EX5//Use mathematical Induction to show :

$1.2.3+2.3.4+\ldots+n(n+1)(n+2)= n(n+1)(n+2)(n+3)/4$

Solution:

Basis step :p(1) is true since

$1.2.3=1(2)(3)(4)/4$

$6=6$

Induction Step: assume p(k) is true

$1.2.3+\ldots+K(k+1)(k+2)=k(k+1)(k+2)(k+3)/4$

$P(k+1)= 1.2.3+\ldots+(k+1)(k+2)(k+3)= (k+1)(k+2)(k+3)(k+4)/4$

We must prove $p(k)\rightarrow p(k+1)$

$1.2.3+\ldots+k(k+1)(k+2)+(k+1)(k+2)(k+3)= k(k+1)(k+2)(k+3)/4+(k+1)(k+2)(k+3)$

$= (k+1)(k+2)(k+3)\{k/4+1\}$

$= (k+1)(k+2)(k+3)(k+4)/4$

EX6//Use mathematical Induction to show :

$\sum_{j=1}^{n} 1/2^j = 2^n-1/2^n$, n=1

Solution:

Basis step :p(1) is true since

$1/2=1/2$

Induction Step: assume p(k) is true

$\sum_{j=1}^{k} 1/2^j = 2^k-1/2^k$

$p(k+1)= \sum_{j=1}^{k+1} 1/2^j = 2^{k+1}-1/2^{k+1}$

We must prove $p(k)\rightarrow p(k+1)$

$\sum_{j=1}^{k} 1/2^j + 1/2^{k+1}$

$=(2^k-1)/2^k +1/2^{k+1}= (2(2^k-1)+1)/2^{k+1}$

$=(2^{k+1}-2+1) / 2^{k+1}$

$=(2^{k+1}-1)/2^{k+1}$

➢ Recursive

Defining the object in terms of itself. Use to define sequence , function, sets.

Recursively Defined Functions

We use two steps to define a function with the set of nonnegative integers as its domain:

BASIS STEP: Specify the value of the function at zero.

RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

Such a definition is called a recursive or inductive definition.

EX//Suppose that f is defined recursively by
f(0)=3
f(n + 1) = 2*f(n) + 3.
Find f(1), f(2), f(3), and f(4).

Solution:

From the recursive definition it follows that
f(1) = 2*f(0) + 3 = 2 * 3 + 3 = 9,

f(2) = 2*f(1) + 3 = 2 *9 + 3 = 21 ,

f(3) = 2*f(2) + 3 = 2 * 21 + 3 = 45 ,

f(4) = 2*f(3) + 3 = 2 * 45 + 3 = 93 .

EX// if f is defined recursively by f(0)=-1, f(1)=2
Find f(2),f(3), f(4), where f(n+1) =f(n)+3f(n-1)
Solution:
F(2)=f(1)+3f(0)
=2+3(-1) =-1
F(3)=f(2)+3f(1)
=-1+3(2) =5
F(4)=f(3)+3f(2)
=5+3(-1) =2

EX// Give an inductive definition of the factorial function F(n) = n!, where F (0) = 1 and find f(5) ?

Solution:
the desired rule is f (n + 1) = (n + 1)f(n).
f(5) = 5 f(4)
= 5 . 4F (3)
= 5 · 4 · 3 F (2)
= 5 · 4 · 3 · 2 F (1)
= 5 · 4 . 3 . 2 . 1 . F(0)
= 5 . 4 . 3 . 2 . 1 . 1 = 120.

EX// Find the Fibonacci numbers f2, f3, f4, f5, and f6?

Solution:
Because the first part of the definition states that f_0= 0 and f_1 = 1, it follows from the second part of the definition that

$f_2 = f_1 + f_0 = 1 + 0 = 1$,

$f_3 = f_2 + f_1 = 1 + 1 = 2$,

$f_4 = f_3 + f_2 = 2 + 1 = 3$,

$f_5 = f_4 + f_3 = 3 + 2 = 5$,

$f_6 = f_5 + f_4 = 5 + 3 = 8$

➢ Methods of Proving Theorems

- **Direct Proofs**

  The implication p →q can show that if p is true then q must be true.
  P true and q false never occurs.
  This kind of proof is called Direct Proofs.

EX// Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution:

N is odd , there exist an integer k such that
n =2k + 1
$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$
$\therefore n^2$ is an odd integer.

- **Indirect Proof**.

  The implication p →q is equivalence to ¬q →¬p , we can proof p → q by showing its **contrapositive** ¬q →¬p is true. This called Indirect Proof.

Ex1// Prove by indirect proof that if n is an integer and 3n + 2 is odd, then n is odd.

Solution:

let n is even , so n=2k , k any integer
3n+2=3(2k)+2
= 2(3k+1)
So 3n + 2 is even, therefore not odd.
So the original conditional statement is true

EX2// prove that if n is an integer and n^2 is odd, then n is odd ?

Solution:

N^2=2k+1
N= $\pm\sqrt{2k+1}$
The direct prop is not useful , We will use the indirect proof
Let n be even number , so there is
N=2k
$N^2=4k^2 =2(2k^2)$
Which implies that n2 is also even, therefore not odd.
So the original conditional statement is true

EX3// Show that these statements about the integer n are equivalent:

P1 : n is even.

P2 : n-1 is odd.

Solution:

We will show P1$\rightarrow$P2

Suppose that n is even. Then n = 2k

$n - 1 = 2k - 1$

$= 2k - 2 + 1$

$= 2(k - 1) + 1$.

This means that n - 1 is odd.

EX4// Is the following argument correct? It supposedly shows that n is an even integer whenever n^2 is an even integer.

Solution:

LET $n = 2k$

$n^2 = 4k^2$

$= 2(2k^2)$

EX5// Show that these statements about the integer n are equivalent:

P1: n -1 is odd.

P2 : n^2 is even.

Solution:

$n - 1 = 2k + 1$

$n = 2k + 2$

$n^2 = (2k + 2)^2$

$=4k^2+8k+4$

$=2(2k^2+4k+2)$

$\therefore N^2$ is even

EX6// prove if x is odd then x+1 is even ?

Solution:

X is odd

X=2k+1

X+1=2k+2

=2(k+1)

∴x+1 is even

EX7// prove that sum of 2 odd numbers is even?

Solution:

Let n,m the 2 odd numbers

N=2k+1

M=2L+1

N+M=2k+2L+2

=2(k+L+1)

∴the sum is even

EX8// Give a proof by contradiction of the theorem "If 3n + 2 is odd, then n is odd."

Solution:

Let n be even

N=2k

3n=6k

3n+2=6k+2

=6k+2

=2(3k+1)

∴3n+2 is even , it is contradiction “3n + 2 is odd”

EX9// Give a proof by contradiction of the theorem "If $n^2 + 2$ is even, then n is even."

Solution:

Let n be odd

$N=2k+1$

$N^2=4k^2+4k+1$

$N^2+2=4k^2+4k+3$

$=4k^2+4k+2+1$

$=2(2k^2+2k+1)+1$

$\therefore n^2 + 2$ is odd , it is contradiction "$n^2 + 2$ is even"

EX10// let $a \in z$ then a^2-a is even, what type a is odd or even ?

Solution:

When a is odd

$a=2k+1$

$a^2=4k^2+4k+1$

$a^2-a =4k^2+4k+1-2k-1$

$=4\ k^2+2k$

$=2(2\ k^2+k)$

A2-a is even

When a is even

$a=2k$

$a^2=4\ k^2$

$a^2-a=4\ k^2-2k$

$=2(2\ k^2-k)$

Also a^2-a is even

CHAPTER FIVE

- Relations
- Properties of Relations
- Operations Relations
- Computer Representation of Relations

• Relations

Relationships between elements of sets occur in many contexts. Every day we deal with. Relationships such as those between a business and its telephone number, The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called **binary relations**.

Definition// Let A and B be sets. A **binary relation** from A to B is a subset of A x B. In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation aRb to denote that (a, b) $\in$ R. Moreover, when (a, b)belongs to R, a is said to be related to b by R.

Ex :Let A = {0, 1 , 2} and B = {a , b } . Then {(0, a), (0, b), (1 , a), (2, b)} is a relation from A to B .
This means, for instance, that 0Ra , but that 1 R/ b. Relations can be represented graphically,

• Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these here.

1. A relation R on a set A is called **identity** if it contains the ordered pair (a, b) $\in$ AXA such that a=b .

Ex: let A= {0, 1, 2, . . . }
R={(0, 0),(1, 1),(2, 2), . . . }

2. A relation R on a set A i s called **reflexive** if (a, a) ∈ R for every element a ∈A.

EX: Consider the following relations on { l , 2, 3 , 4} :
R1 = {(1 , 1), (1 , 2), (2 , 1), (2, 2), (3 , 4), (4, 1) , (4, 4)} ,
R2 = {(1 , 1), (1 , 2), (2, 1) } ,
R3 = { (1 , 1) , (1 , 2) , (1 , 4), (2, 1), (2, 2), (3 , 3) , (4, 1) , (4, 4)},
R4 = {(2, 1), (3 , 1) , (3 , 2), (4, 1) , (4, 2), (4, 3)} ,
R5 = {(1 , 1) , (1 , 2), (1 , 3), (1 , 4), (2, 2), (2, 3), (2, 4), (3 , 3), (3 , 4), (4, 4)} ,
R 6 = {(3 , 4)} .
Which of these relations are reflexive?

Solution: The relations R3 and R5 are reflexive because they both contain all pairs of the form (a , a), namely, (1 , 1), (2, 2), (3 , 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R1 , R2 , R4, and R6 are not reflexive because (3 , 3) is not in any of these relations.

3. A relation R on a set A is called **symmetric** if(b, a) ∈ R whenever (a, b) ∈ R, for all a , b ∈A

EX// Which of the relations in previous example are symmetric?
Solution: The relations R2 and R3 are symmetric, because in each case (b , a) belongs to the relation whenever (a , b) does. For R2 , the only thing to check is that both (2, 1) and (1 , 2) are in the relation. For R3 , it is necessary to check that both (1 , 2) and (2, 1) belong to the relation, and (1 , 4) and (4, 1) belong to the relation. You should verify that none of the other relations is symmetric. This is done by finding a pair (a , b) such that it is in the relation but (b, a) is not.

4. A relation R on a set A is called **transitive** if whenever (a , b) ∈ R and (b, c) ∈ R , then (a , c) ∈ R , for all a , b , c ∈ A.

EX// Which of the relations in previous example are transitive?
Solution: R4 , R5 , and R6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a , b) and (b, c) belong to this relation, then (a , c)

also does. For instance, R4 is transitive, because (3 , 2) and (2 , 1), (4, 2) and (2 , 1), (4, 3) and (3 , 1), and (4, 3) and (3 , 2) are the only such sets of pairs, and (3 , 1), (4, 1), and (4, 2) belong to R4 . You should verify that R5 and R6 are transitive. R1 is not transitive because (3 , 4) and (4, 1) belong to R I , but (3 , 1) does not. R2 is not transitive because (2 , 1) and (1 , 2) belong to R2 , but (2 , 2) does not. R3 is not transitive because (4, 1) and (1 , 2) belong to R3 , but (4, 2) does not. ...

5. A relation R on a set A to B is called **inverse** relation denoted by R^{-1} from B to A contains the ordered pair (b, a) such that (a, b) R.

Ex: R = { (1, 3), (1, 2), (2, 3)} at A={1, 2, 3}
R^{-1} ={(3, 1), (2, 1), (3, 2)}

- **Operations Relations**
  - **Combining Relations**

Because relations from A to B are subsets of A x B, two relations from A to B can be combined in any way two sets can be combined.

EX1// Let A = { l , 2 , 3 } and B = { 1 , 2 , 3 , 4 } , the relations R 1 = { (1 , 1), (2 , 2), (3 , 3)} and R2 = { (1 , 1), (1 , 2), (1 , 3), (1 , 4)}, then can be combined to obtain :
R 1U R2 = {(1 , 1) , (1 , 2), (1 , 3), (1 , 4) , (2 , 2), (3 , 3) } ,
R 1 ∩ R2 = { (1 , 1) } ,
R 1- R2 = {(2 , 2), (3 , 3) } ,
R2 - R I = { (1 , 2) , (1 , 3), (1 , 4)} .

EX2// Let A be the set of all students and B the set of all courses. Suppose that R1 consists of all ordered pairs (a , b), where a is a student who has **taken** course b, and R2 consists of all ordered pairs (a , b), where a is a student who **requires** course b to graduate. **What are the relations R1 U R2 , R1 ∩ R2 , R1 - R2 , and R2 – R1?**
Solution: The relation R1 U R2 consists of all ordered pairs (a , b), where a is a student who **either** has taken course b or needs course b to graduate, and

R1 ∩ R2 is the set of all ordered pairs (a , b), where a is a student who has taken course b **and** needs this course to graduate.
R 1 - R2 is the set of ordered pairs (a , b), where a has taken course b **but does not** need it to graduate; that is, b is an elective course that a has taken.
R2 – R1 is the set of all ordered pairs (a , b), where b is course that a needs to graduate **but has not** taken.

- **Composite relations**

R and S is the relation consisting of ordered pairs (a , c), where a ∈ A, c ∈ C , and for which there exists an element b ∈ B such that (a , b) ∈ R and (b , c) ∈ S. We denote the composite of R and S by S 0 R .
**Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation.

EX// What is the composite of the relations R and S, where R is the relation from {1, 2, 3} to { 1 , 2, 3, 4} with R ={(1 , 1) , (1 , 4), (2, 3), (3 , 1) , (3 , 4)} and S is the relation from { l , 2, 3, 4} to {0, 1 , 2} with S = {(1 , 0), (2, 0), (3 , 1) , (3 , 2), (4, 1)}?

Solution: S 0 R is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. where

S o R = {(1 , 0), (1 , 1) , (2, 1) , (2, 2), (3 , 0), (3 , I) } .

- **Equivalence Relations**

A relation on a set A is called an **equivalence** relation if it is **reflexive**, **symmetric**, and **transitive**.

Ex: Let A={1, 2, 3, 4} and R={(1, 1), (2, 2), (3, 3), (4, 4), (2, 4), (4, 2)}
Solution:

R is reflexive, R is symmetric, R is transitive

∴ R is equivalent relation.

➢ Computer Representation of Relations

1. Representing Relations Using Matrices

A relation can be represented using a zero-one matrix

$$M(i, j)=\begin{cases}1 \text{ if } (a_i, b_j) \text{ in } R\\ 0 \text{ if } (a_i, b_j) \text{ not in } R\end{cases}$$

Ex// Suppose that A = { 1, 2 , 3 } and B = { 1 , 2 } . Let R be the relation from A to B containing (a , b) if a ∈ A, b ∈ B , and a > b. What is the matrix representing R if al = 1 , a2 = 2, and a3 = 3 , and bl = 1 and b2 = 2 ?

Solution: Because R = {(2, 1) , (3 , 1) , (3 , 2)}, the matrix for R is:

$$M_R = \begin{bmatrix}0 & 0\\ 1 & 0\\ 1 & 1\end{bmatrix}$$

The 1's in MR show that the pairs (2 , 1), (3 , 1), and (3 , 2) belong to R . The 0's show that no other pairs belong to R

Ex// Let A = {a1 , a2 , a3 } and B = {bl , b2 , b3 , b4 , b5 } . Which ordered pairs are in the relation R represented by the matrix

$$MR=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 1 & 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\end{bmatrix}$$

Solution:

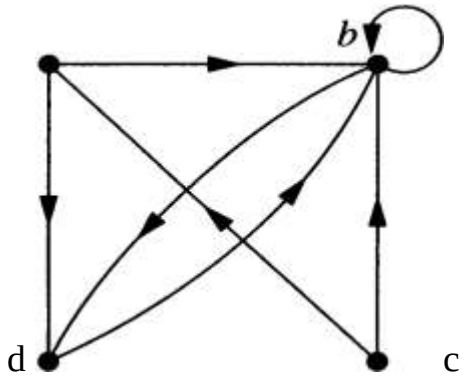
Because R consists of those ordered pairs (ai , bj) with m ij = 1 ,
it follows that
R = {(a1 , b2), (a2 , bl), (a2 , b3), (a2 , b4), (a3 , b1), (a3 , b3), (a3 , b5)}.

2. Representing Relations Using graphs

Each element of the set is represented by a point and each ordered pair is represented by arc with its direction.

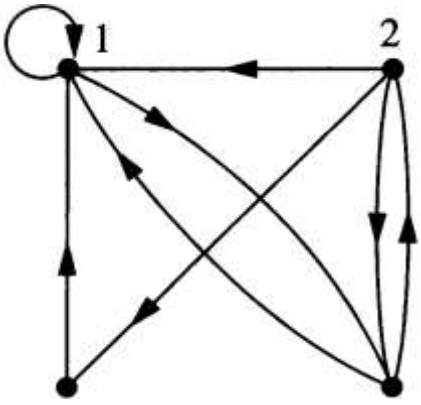
Ex: Let A={a, b, c, d},
R={(a, b), (a, d), (b, b),(b, d), (c, a),(c, b), (d, b)}, Which ordered pairs are in the relation R represented by the direct graph?

Solution:



EX: Let A={1, 2, 3, 4},
R={ (1 , 1) , (1 , 3), (2 , 1), (2, 3), (2 , 4) , (3 , 1) , (3 , 2), (4 , 1) }, Which ordered pairs are in the relation R represented by the direct graph?

Solution:



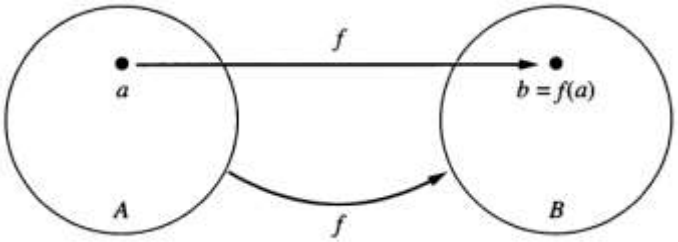
CHAPTER SIX

- **Functions**
- **Domain and codomain of the function**
- **Range of the function**
- **Graph of function**
- **Functions types**

➢ Functions

DEF// Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A . We write f(a) = b if b is the unique element of B assigned by the function f to the element a of A . If f is a function from A to B, we write $f : A \rightarrow B$.

$\forall a \in A, \exists b \in B: (a,b) \in f$



➢ Domain and Codomain of the function

Let $f: A \rightarrow B$ is function

A is called the Domain of f denoted by dom f

dom f =A

B is called the codomain of f denoted by cod f

Cod f =B

➢ Range of the function

Let $f:A \rightarrow B$ is function

The range of f is the set of all images of elements of A is in B

Range f = { $b: b \in B, \forall a \in A \rightarrow b=f(a)$}

Range $f \subseteq B$

- **Graph of function**

Let f be a function from A to B , the graph of f is the set of ordered pairs (a,b) such that {(a,b) | a∈A and b= f(a)}

EX1// let f: R→ R

$Y=f(x)=x^2$

Find Dom, Cod , Rang, Graph(f) ?

Solution:

Dom f = R

Cod f = R

Rang f = x^2= {1,4,9,….}

Graph(f) = {(1,1),(2,4),…,(-1,1),(-2,4),…}

EX2// let f: Z+→ Z+

f(x)=10x+1

Find Dom, Cod , Rang, Graph(f) ?

Solution:

Dom f = Z+

Cod f = Z+

Rang f ={1,11,21,...}

Graph(f) ={(x, 10x+1)}= {(0,1),(1,11),(2,21),…}

- **Functions types**

1. injection function (one-to-one)

A function f is said to be **one-to-one, or injective**, if and only if f(a) = f(b) implies that a = b for all a and b in the domain of f. A function is said to be an injection if it is one-to-one.

EX1// Let A={a,b,c}, B={1,2,3,4}, F={(a,1),(b,2),(c,3)}

Is f an injective function?

Solution:

∴ f is injective

EX2// Let $f: R \rightarrow R$, $\forall x \in R$ $f(x)= x^2$, Is f an injective function?

Solution:

$2 \in R$ $f(2)=4$

$-2 \in R$ $f(-2)=4$

$\therefore$ f is not injective functions

2- **Surjective function(onto)**

A function f from A to B is called **onto, or surjective**, if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$. A function f is called a surjection if it is onto.

A function f is onto if $\forall y \exists x\ (f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

EX//

A={a,b,c,d}, B={1,2,3}

F={(a,1),(b,1),(c,2),(d,2)}

Solution:

Rang f ={1,2}$\neq$ B

$\therefore$ f not onto

3- **Bijection function**

The function f is said to be bijection , if it is both injection (one-to-one) and surjection (onto).

Ex//

f: A$\rightarrow$ B, A={a,b,c,d}, B={1,2,3,4}

F(a)=4 , F(b)=2, F(c)=1 , F(d)=3

Solution:

$\therefore$ f **bijection,** because is injection (one-to-one) and surjection (onto).

4- **Inverse function**

Let f be one-to-one and onto from A→B , the inverse function (f^{-1}) of f such that f(a)=b → $f^{-1}(b)=a$

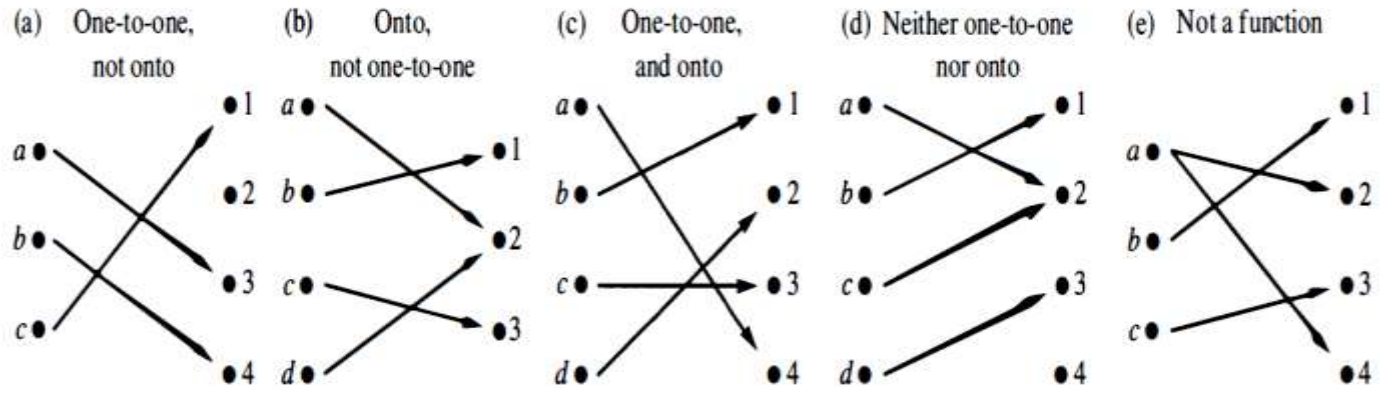
EX// Let f be the function from {a , b, c} to { l , 2, 3} such that f(a) = 2, f(b) = 3, and f(c) = 1 . Is f invertible, and if it is, what is its inverse?

Solution:

The function f is invertible because it is a one-to-one and onto correspondence.
The inverse function f^{-1} reverses the correspondence given by f, so
f^{-1} (1) = c, f^{-1} (2) = a, and f^{-1} (3) = b.

***Types of function Using graphs**

Let f be the function from {a , b, c} to { l , 2, 3,4}



H.W//

Let f be the function f:R→R and $f(x)=x^3$

Find

Is f injection?

Is f surjection?

Is f bijection?

Is f inverse?

CHAPTER SEVEN

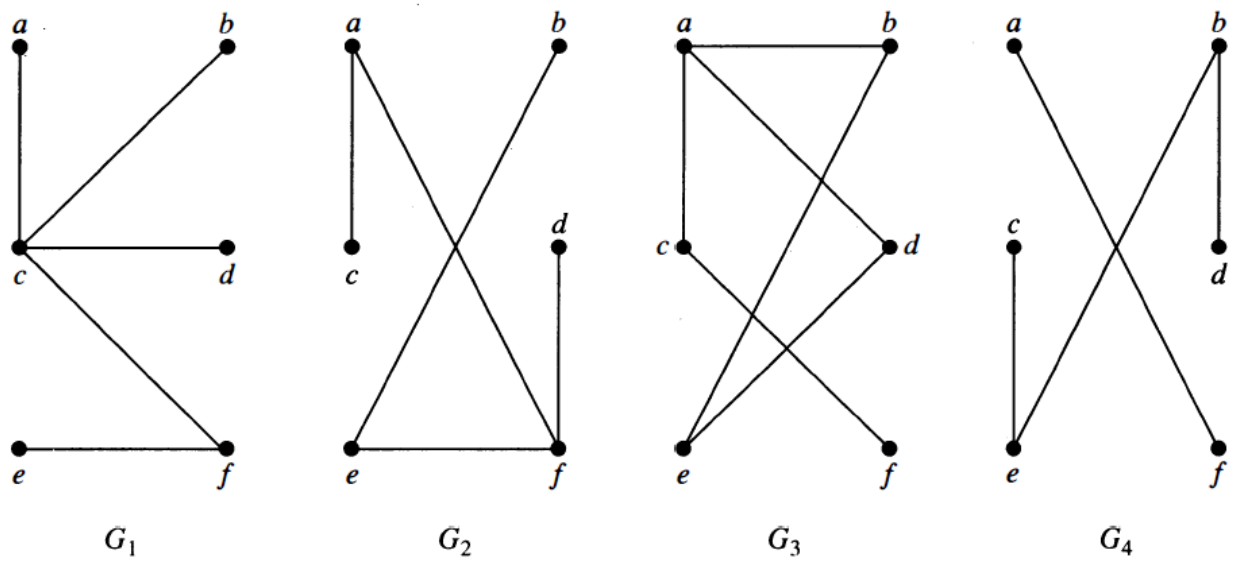
- Trees
- Trees as Models
- Types of trees
- Properties of Trees
- Universal Address Systems
- Tree Traversal
- Traversal Algorithms
- Infix, Prefix, and Postfix Notation

- **Tree**

Definition// A tree is a connected undirected graph with no simple circuits.

EX// Which of the graphs shown in following Figure are trees?

Solution:
G1 and G2 are trees, because both are connected graphs with no simple circuits. G3 is not a tree because e, b, a, d, e is a simple circuit in this graph. Finally, G4 is not a tree because it is not connected.



Definition// A **rooted tree** is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

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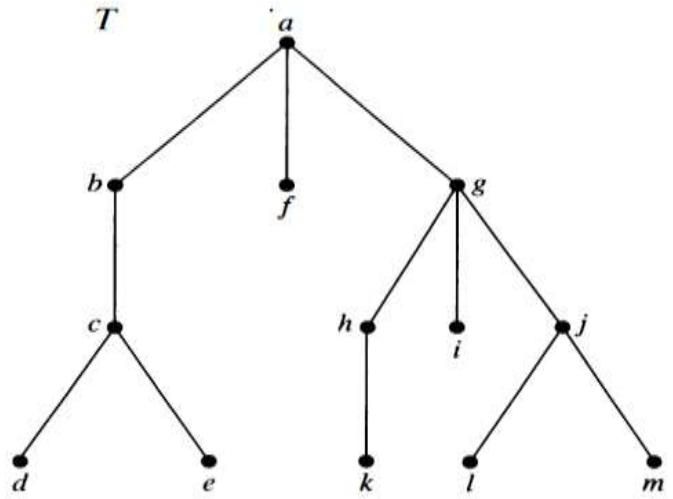


- Parent of b= a
- Children of g= h,i,j
- Siblings of h =i,j
- All ancestors of e =c,b,a
- All descendants of b= c,d,e
- internal vertices= a,b,c,g,h,j
- all leaves= d,e,f,k,i,l,m

- **Types of trees**

1. m -ary tree

Definition// A rooted tree is called an m -ary tree if every internal vertex has no more than m children.

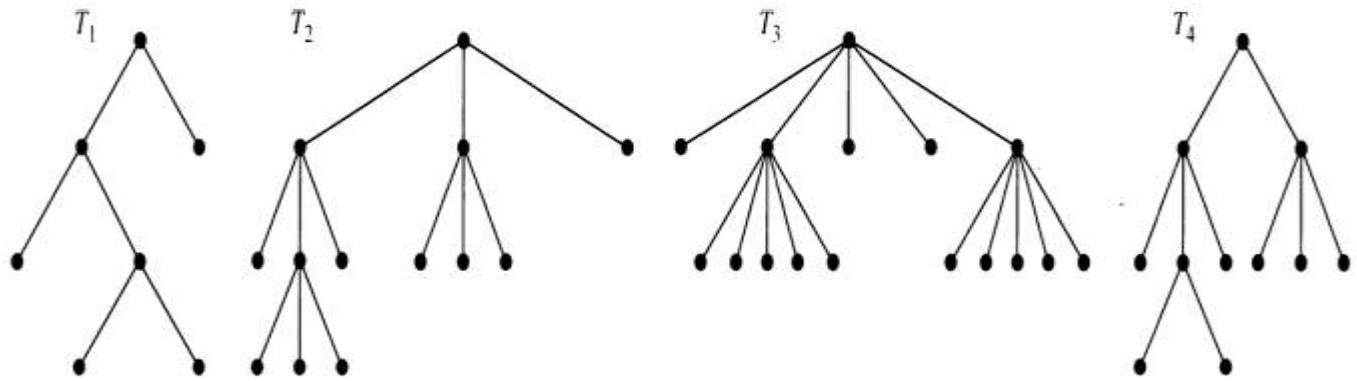
2. full m –ary tree

Definition// The tree is called a full m -ary tree if every internal vertex has exactly m children.

3. binary tree

Definition// An m –ary tree with m = 2 is called a binary tree.

EX//Are the rooted trees in following Figure full m -ary trees for some positive integer m ?



Solution:

Tl i s a full binary tree because each of its internal vertices has two children. T2 is a full 3-ary tree because each of its internal vertices has three children. In T3 each internal vertex has five children, so T3 is a fu1l 5-ary tree. T4 is not a full m -ary tree for any m because some of its internal vertices have two children and others have three children.

- **Trees as Models**

Trees are used as models in such diverse areas as computer science, chemistry, geology, botany, and psychology. We will describe a computer models based on trees.

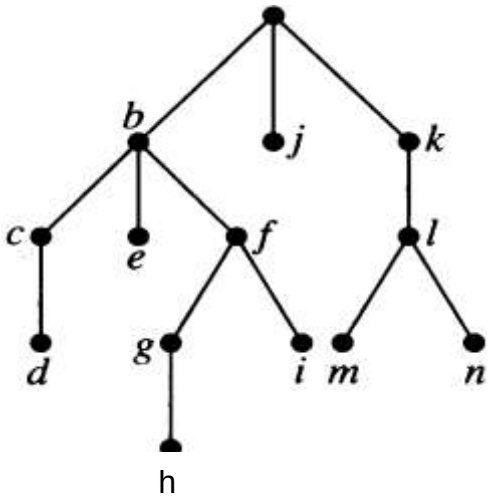
Computer File Systems Files in computer memory can be organized into directories. A directory can contain both files and subdirectories.
The root directory contains the entire file system. Thus, a file system may be represented by a rooted tree, where the root represents the root directory, internal vertices represent subdirectories, and leaves represent ordinary files or empty directories.

- **Properties of Trees**

We will often need results relating the numbers of edges and vertices of various types in trees.

1- A tree with n vertices has n - 1 edges.
2- The level of vertex in rooted tree is the length of the unique path from root to the vertex, where the root level is zero.
3- The height of rooted tree is the maximum of the level of vertices.

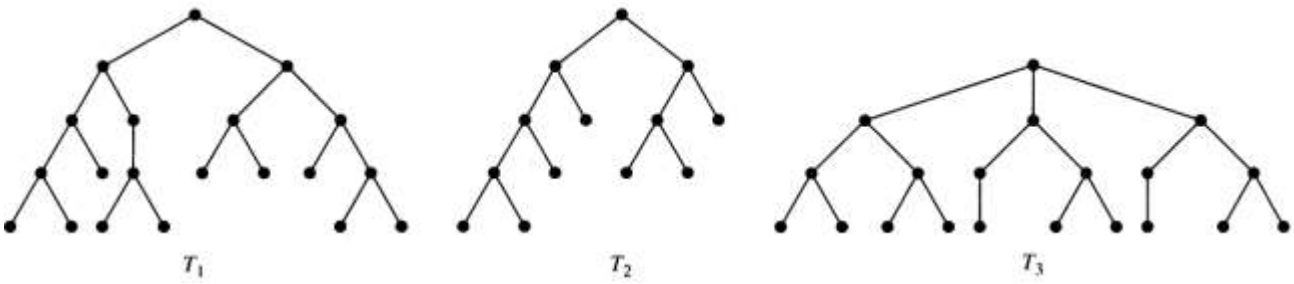
EX// Find the level of each vertex in the rooted tree shown in following Figure. What is the height of this tree?



Solution:
The root a is at level 0.
Vertices b, j, and k are at level 1 .
Vertices c, e, f, and l are at level 2.
Vertices d, g, i , m , and n are at level 3 .
Finally, vertex h is at level 4. Because the largest level of any vertex is 4, this tree has height 4.

Definition// A rooted m -ary tree of height h is **balanced** if all leaves are at levels h or h - 1 .

EX// Which of the rooted trees shown in flows Figure are balanced?



Solution:
Tl is balanced, because all its leaves are at levels 3 and 4.
However, T2 is not balanced, because it has leaves at levels 2, 3, and 4.
Finally, T3 is balanced, because all its leaves are at level 3 .

- **Tree Traversal**

Ordered rooted trees are often used to store information. We need procedures for visiting each vertex of an ordered rooted tree to access data. We will describe several important algorithms for visiting all the vertices of an ordered rooted tree. Ordered rooted trees can also be used to represent various types of expression.
The different listings of the vertices of ordered rooted trees used to represent expressions are useful in the evaluation of these expressions.

- **Universal Address Systems**

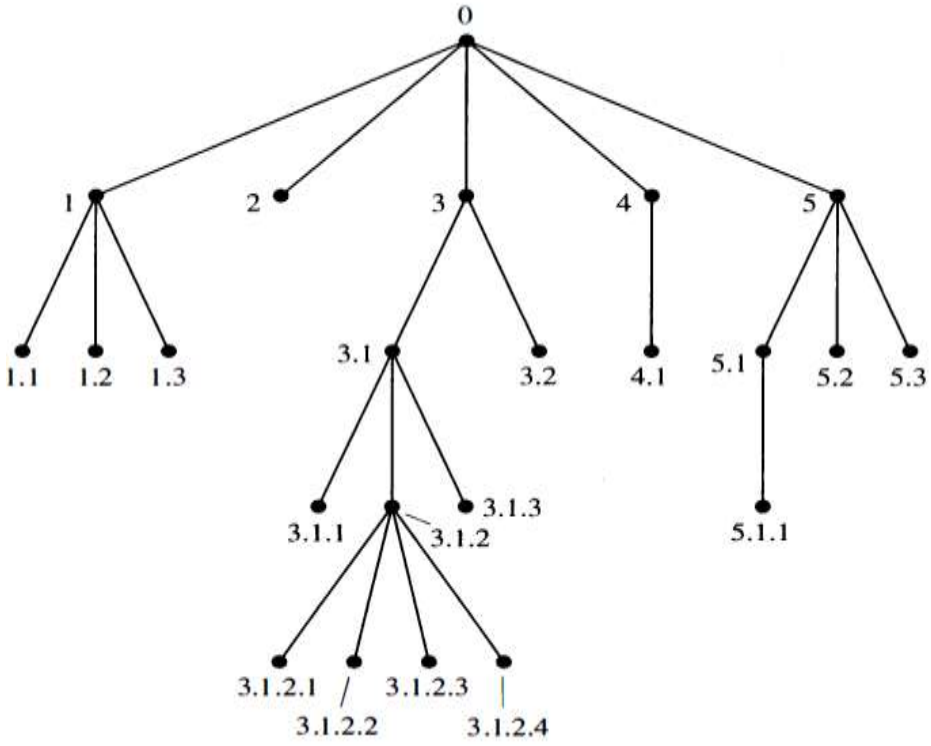
Procedures for traversing all vertices of an ordered rooted tree rely on the orderings of children. In ordered rooted trees, the children of an internal vertex are shown from **left to right** in the drawings representing these directed graphs.

To produce this ordering, we must first label all the vertices. We do this recursively:

1 . Label the root with the integer 0. Then label its k children (at level 1) from left to right with 1, 2, 3 , . . . , k.

2. For each vertex v at level n with label A, label its kv children, as they are drawn from left to right, with A . 1 , A .2, . . . , A .kv .

EX// We display the labelings of the universal address system next to the vertices in the ordered rooted tree shown in following Figure . what **the lexicographic ordering of the labelings is?**



Solution:

The lexicographic ordering of the labelings is?

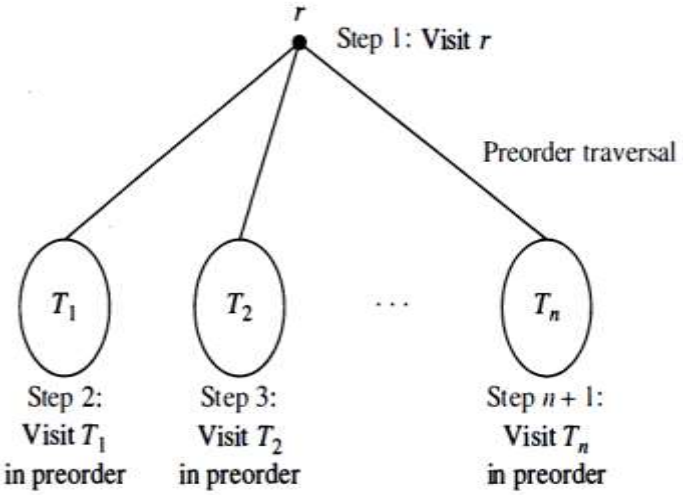
0 < 1 < 1 . 1 < 1 .2 < 1 .3 < 2 < 3 < 3 . 1 < 3 . 1 . 1 < 3 . 1 .2 < 3 . 1 .2 . 1 < 3 . 1 .2.2 < 3 . 1 .2 . 3 < 3 . 1 .2.4 < 3 . 1 .3 < 3 .2 < 4 < 4. 1 < 5 < 5 . 1 < 5 . 1 . 1 < 5 . 2 < 5 . 3

Traversal Algorithms

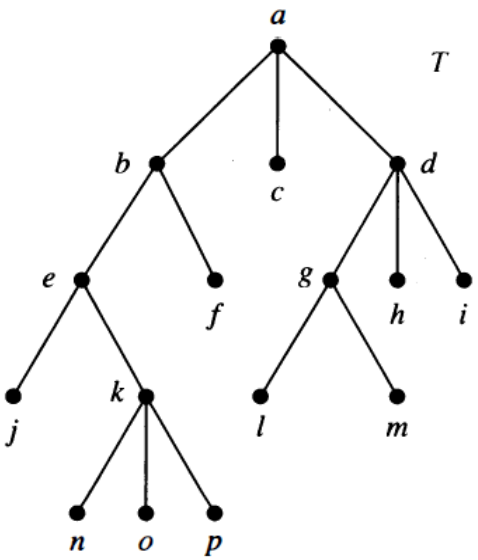
Procedures for systematically visiting every vertex of an ordered rooted tree are called traversal algorithms. We will describe three of the most commonly used such algorithms, preorder traversal, inorder traversal, and postorder traversal. Each of these algorithms can be defined recursively. We first define preorder traversal.

Definition// Let T be an ordered rooted tree with root r . If T consists only of r, then r is the preorder traversal of T . Otherwise, suppose that TJ , Tz , . . . , Tn are the subtrees at r from left to right in T . The preorder traversal begins by visiting r . It

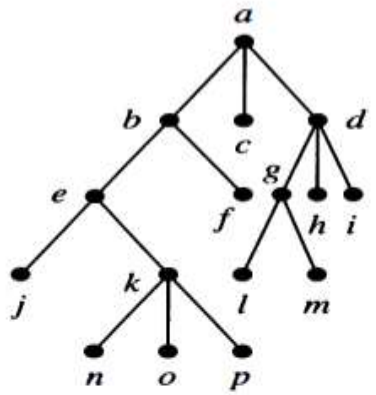
continues by traversing TJ in preorder, then Tz in preorder, and so on, until Tn is traversed in preorder.



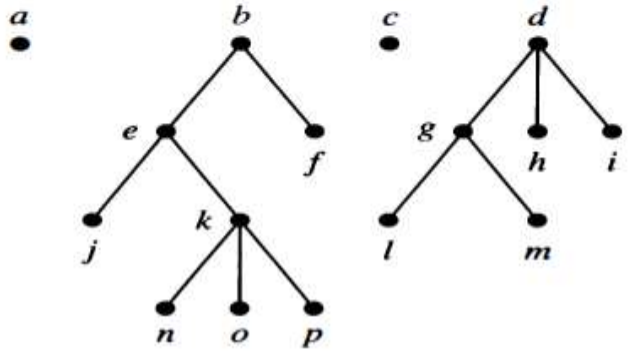
EX// In which order does a preorder traversal visit the vertices in the ordered rooted tree T shown in following Figure ?

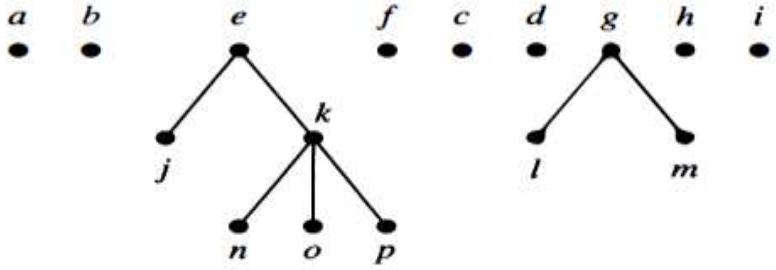


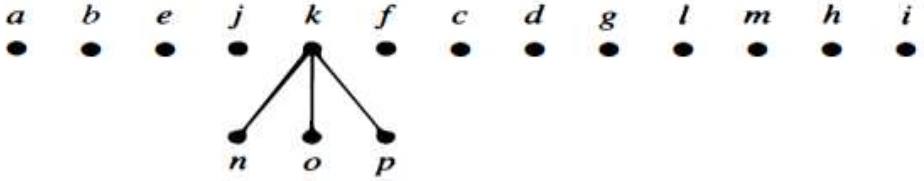
Solution: The steps of the preorder traversal of the ordered rooted tree T are shown as:

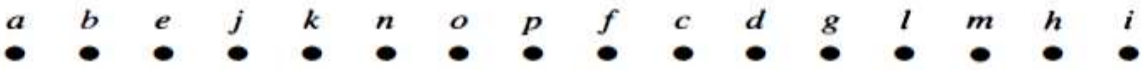


Preorder traversal: **Visit root, visit subtrees left to right**



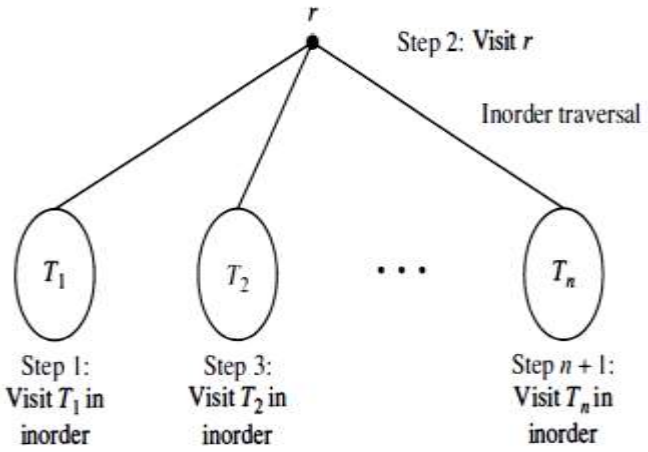




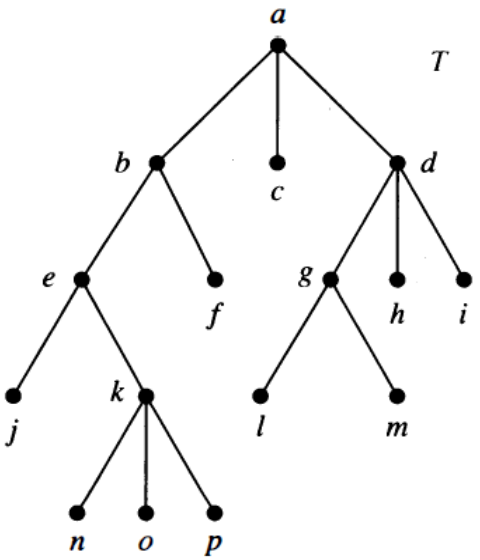


∴ preorder traversal of T is a, b, e, j , k, n , o, p, f, c, d, g, l, m , h , i .

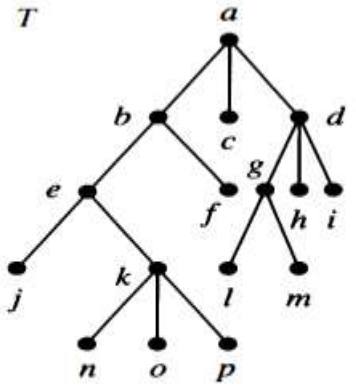
Definition// Let T be an ordered rooted tree with root r. If T consists only of r, then r is the inorder traversal of T . Otherwise, suppose that T1 , T2 , . . . , Tn are the subtrees at r from left to right. The inorder traversal begins by traversing T1 in inorder, then visiting r . It continues by traversing T2 in inorder, then T3 in inorder, . . . , and finally Tn in inorder



EX// In which order does an inorder traversal visit the vertices of the ordered rooted tree T in Figure ?



Solution: The steps of the inorder traversal of the ordered rooted tree T are shown as:



Inorder traversal: Visit leftmost subtree, visit root, visit other subtrees left to right

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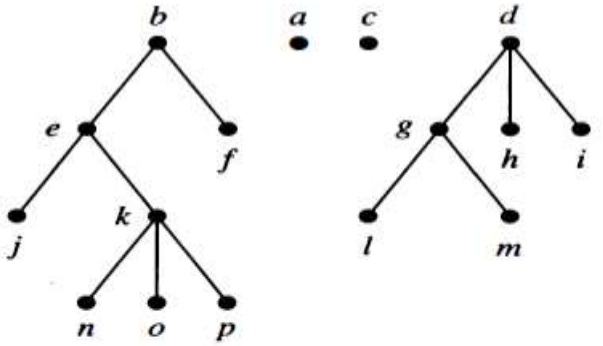
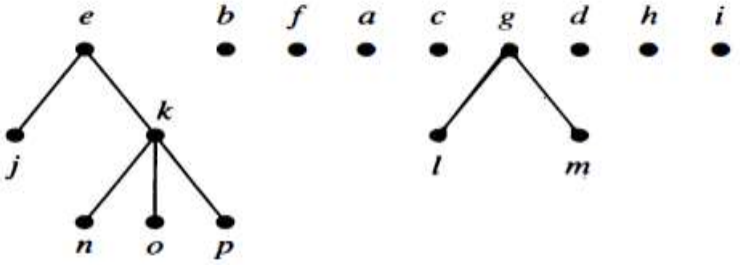
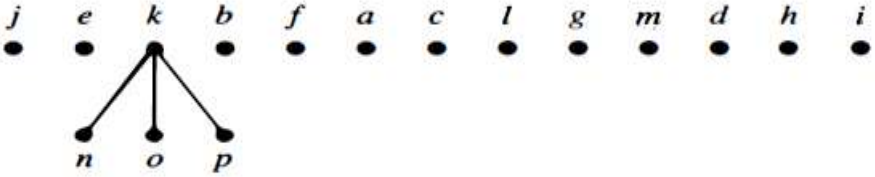
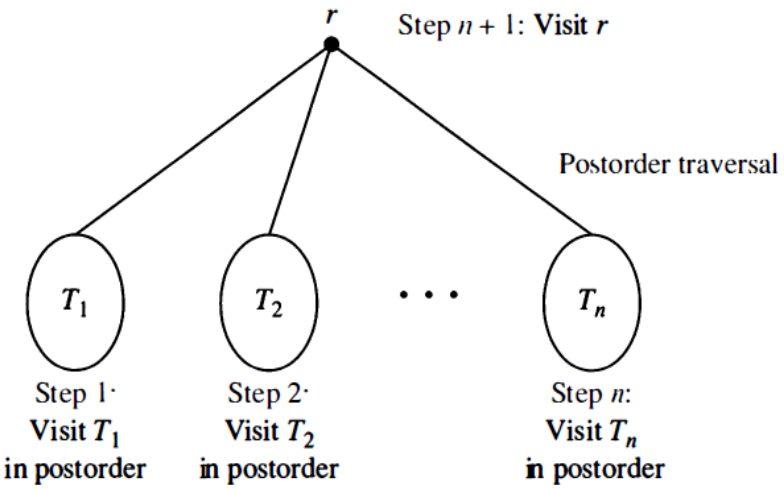




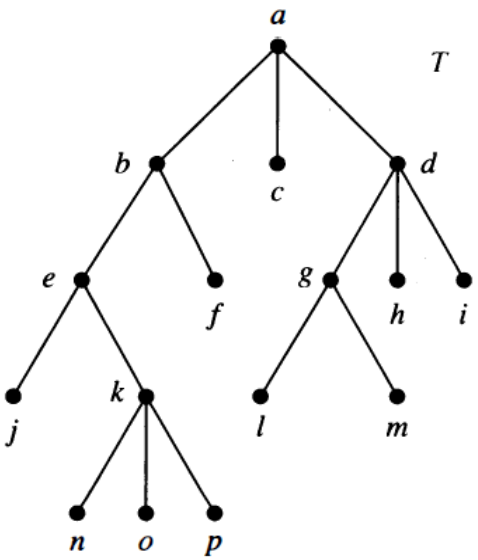


∴ the inorder listing of the ordered rooted tree is j, e, n, k, o, p, b, f, a, c, l, g, m ,d, h, i

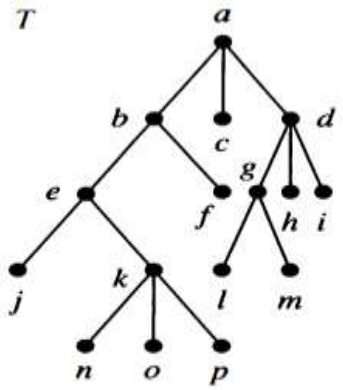
Definition// Let T be an ordered rooted tree with root r . If T consists only of r, then r is the postorder traversal of T . Otherwise, suppose that T1 , T2 , . . . , Tn are the subtrees at r from left to right. The postorder traversal begins by traversing T1 in postorder, then T2 in postorder, . . . , then Tn in postorder, and ends by visiting r .



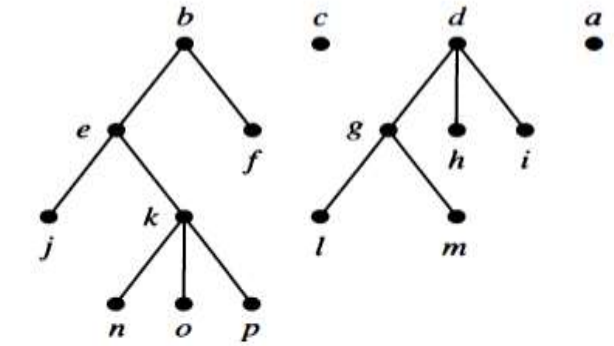
EX// In which order does an postorder traversal visit the vertices of the ordered rooted tree T in Figure ?

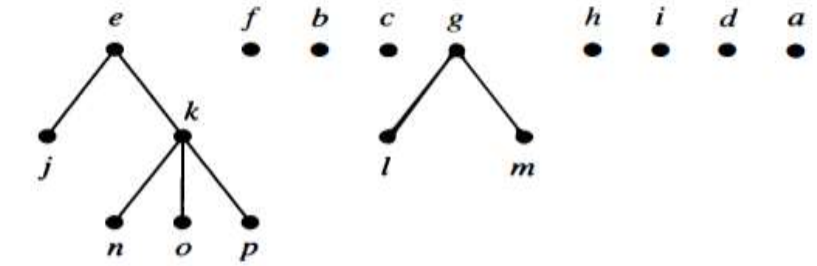


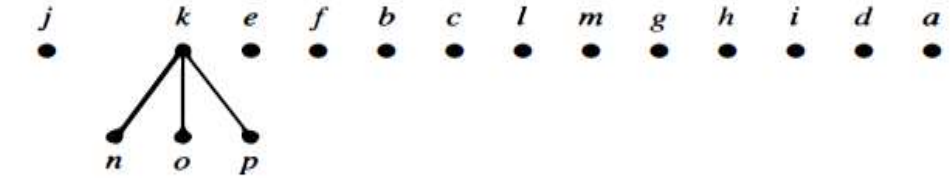
Solution: The steps of the postorder traversal of the ordered rooted tree T are shown as:

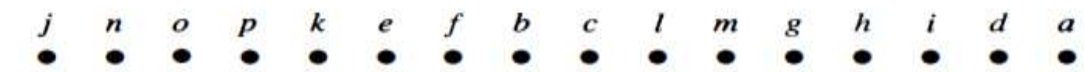


Postorder traversal: Visit subtrees left to right; visit root









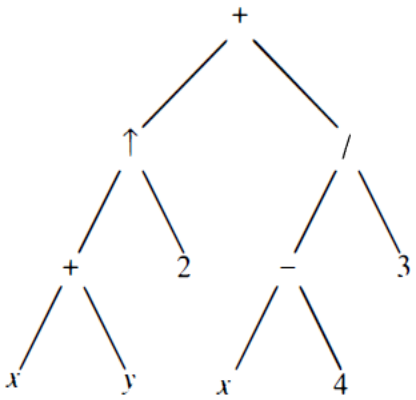
∴ the postorder traversal of T is j , n , o, p, k, e, f, b, c, l, m , g, h , i , d, a

- **Infix, Prefix, and Postfix Notation**

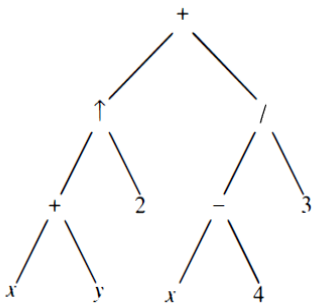
We can represent complicated expressions, such as compound propositions, combinations of sets, and arithmetic expressions using ordered rooted trees. For instance, consider the representation of an arithmetic expression involving the operators + (addition), - (subtraction), * (multiplication), / (division), and↑(exponentiation). We will use parentheses to indicate the order of the operations. An ordered rooted tree can be used to represent such expressions, where the internal vertices represent operations, and the leaves represent the variables or numbers.
Each operation operates on its left and right subtrees (in that order).

EX// What is the ordered rooted tree that represents the expression
((x + y)↑ 2) + ((x - 4)/3)

Solution: These steps are shown in following Figure:

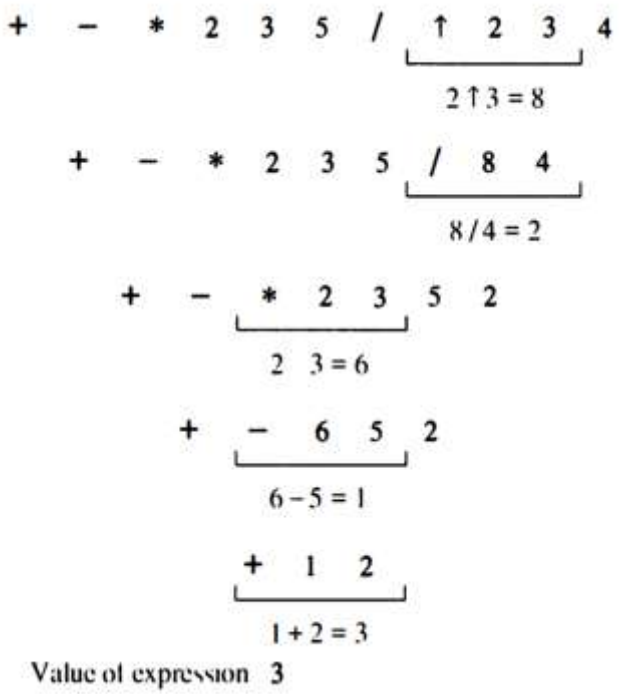


EX// What is the prefix form for ((x + y) ^ 2) + ((x - 4) /3)?
Solution:
We obtain the prefix form for this expression by traversing the binary tree that represents it, shown in following Figure . This produces + ^ + x y 2 / - x 4 3 .



EX// What is the value of the prefix expression + - * 2 3 5/ ^ 2 3 4?

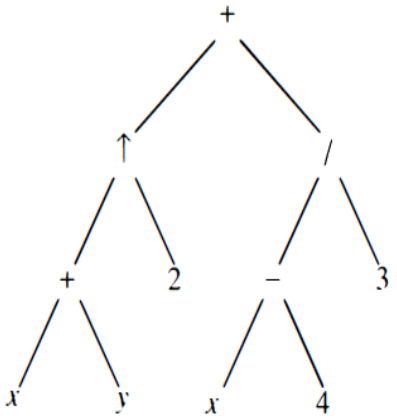
Solution: The steps used to evaluate this expression by working right to left, and performing operations using the operands on the right, are shown in following Figure . The value of this expression is 3 .



EX//What is the postfix form of the expression ((x + y) ^ 2) + ((x - 4) /3)?

Solution:

The postfix form of the expression is obtained by carrying out a postorder traversal of the binary tree for this expression, shown in following Figure .



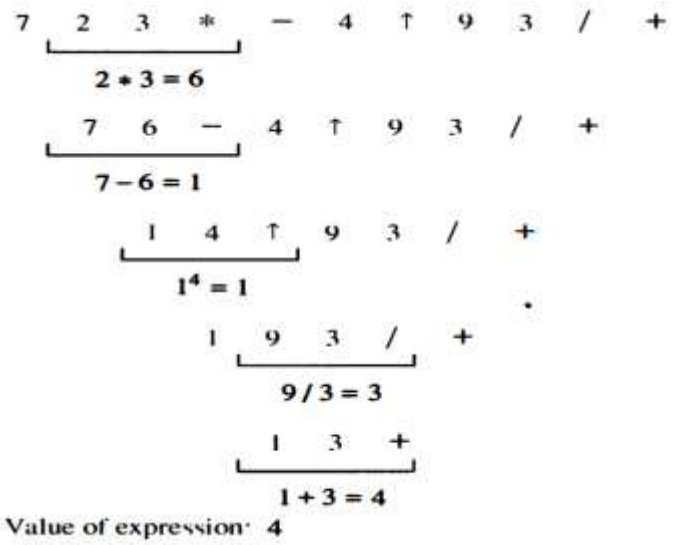
This produces the postfix expression: x y + 2 ^ x 4 - 3 / +.

EX// What is the value of the postfix expression 7 2 3 * - 4 ^ 9 3 / + ?

Solution:

The steps used to evaluate this expression by starting at the left and carrying out operations when two operands are followed by an operator are shown in Figure .

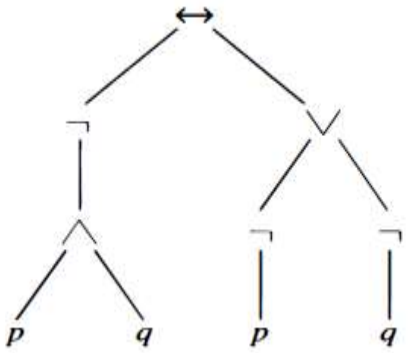
The value of this expression is 4.



EX// Find the ordered rooted tree representing the compound proposition $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$. Then use this rooted tree to find the prefix, postfix, and infix forms of this expression.

Solution:

The rooted tree for this compound proposition is constructed from the bottom up. First, subtrees for $\neg p$ and $\neg q$ are formed (where $\neg$ is considered a unary operator). Also, a subtree for $p \wedge q$ is formed. Then subtrees for $\neg(p \wedge q)$ and $(\neg p) \vee (\neg q)$ are constructed. Finally, these two subtrees are used to form the final rooted tree. The steps of this procedure are shown in following Figure



The prefix, postfix, and infix forms of this expression are found by traversing this rooted tree in preorder, postorder, and inorder (including parentheses), respectively.
These traversals of preorder give $\leftrightarrow \neg \wedge pq \vee \neg p \neg q$,
These traversals of postorder give $pq \wedge \neg p \neg q \neg \vee \leftrightarrow$, and
These traversals of inorder give $(\neg (p \wedge q)) \leftrightarrow ((\neg p) \vee (\neg q))$.

H.W//

1- Represent the expression (x+x*y)+(x/y) using an ordered rooted tree

2- Represent the expression (A∩B)-(A∪(B-A) using an ordered rooted tree

3- What is the value of each of the prefix expression
   a) - * 2 / 8 4 3
   b) ^ - * 3 3 * 4 2 5
   c) + - ^ 3 2 ^ 2 3 / 6 – 4 2
   d) * + 3 + 3 ^ 3 + 3 3 3

4- What is the value of these postfix expression
   a) 5 2 1 - - 3 1 4 + + *
   b) 9 3 / 5 + 7 2 - *
   c) 3 2 * 2 ^ 5 3 – 8 4 / * -

5- Draw the ordered rooted tree to each expression written in prefix notation , then write each one using infix notation
   a) + * + - 5 3 2 1 4
   b) ^ + 2 3 – 5 1
   c) */ 9 3 + * 2 4 – 7 6

CHAPTER EIGHT

- **Graph**
- **The types of graphs**
- **Some Special Simple Graphs**
- **Representing Graphs**
- **Isomorphism and Isomorphic of graphs**

• Graph

Graphs are discrete structures consisting of non-empty set of nodes (Vertices) and a set E of edges that connect pairs of nodes.

• The types of graphs

Different types of graphs have different definitions depending on what kind of edges that used.

1. Endpoints graph

A graph G = (V , E) consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints.

2. infinite and finite graph

A graph with an infinite vertex set is called an infinite graph, and in comparison, a graph with a finite vertex set is called a finite graph.

3. simple graph

A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

4. multigraphs

Graphs that may have multiple edges connecting the same vertices are called multigraphs.

5. pseudographs

Graphs that may include loops (edges that connect a vertex to itself), and possibly multiple edges connecting the same pair of vertices, are sometimes called pseudographs.

6. directed graph

A directed graph (or digraph) (V , E) consists of a nonempty set of vertices V and a set of directed edges E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u , v) is said to start at u and end at v.

7. simple directed graph

When a directed graph has no loops and has no multiple directed edges, it is called a simple directed graph

8. directed multigraphs

Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex, then we called directed multigraphs.

9. mixed graph

a graph where some edges are undirected, while others are directed. A graph with both directed and undirected edges is called a mixed graph.

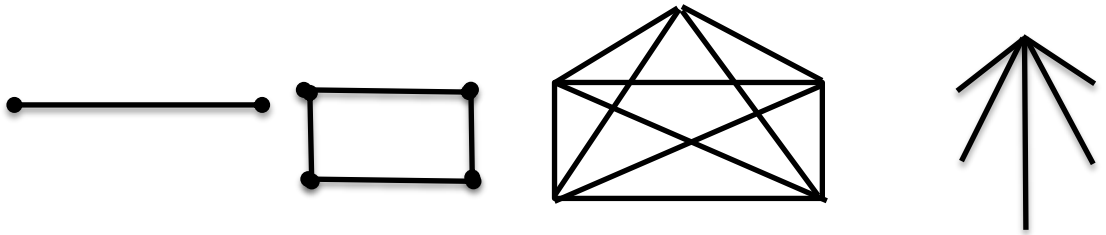
This terminology for the various types of graphs is summarized in Table 1

TABLE 1 Graph Terminology.

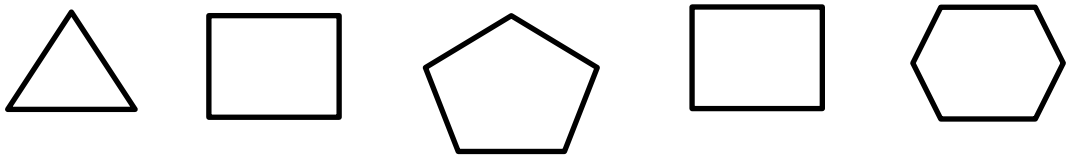
Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

- **Some Special Simple Graphs**

1. Complete graphs: The simple graph that contains exactly one edge between each pair of distinct vertices

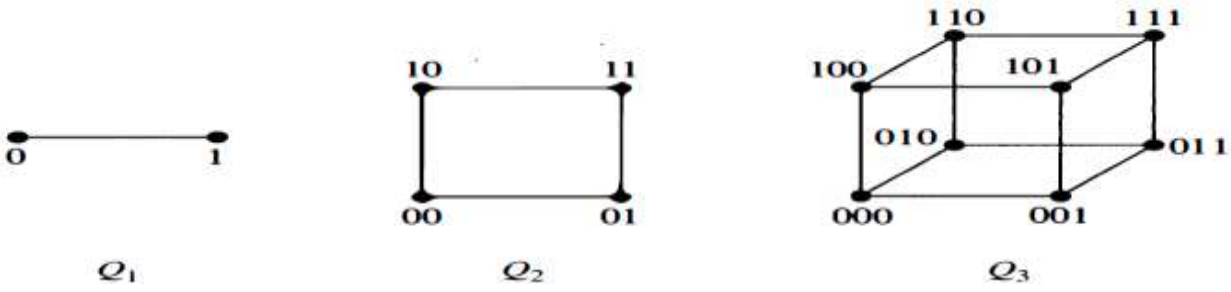


2. Cycles graphs: consists of n vertices v1…..vn and edges (v1,v2), (v2, v3)…(v_{n-1}, vn)



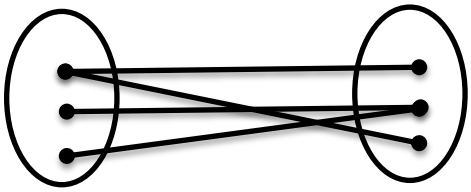
3. wheel graphs: We obtain the wheel when we add an additional vertex to cycle and connect it with each vertex.

4. n- Cubes graphs : It is the graph that has vertices representing the 2^n bit string.



5. Bipartite

A simple graph G is called bipartite if its vertex set V can be partitioned in to 2 disjoint sets v1 and v2 such that every edge in the graph connect a vertex in v1 and a vertex in v2



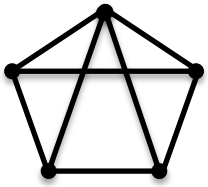
Complete Bipartite Graph:

K23

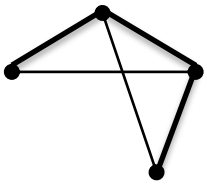
K26

6. Subgraph

A subgraph of a graph G=(V,E) is a graph, H=(W,F) , where W⊆V and F⊆E

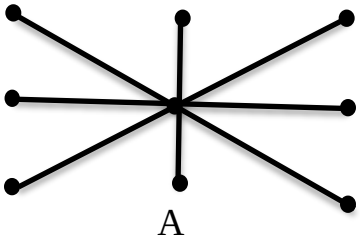


G

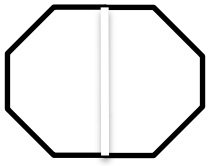


H

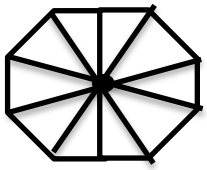
- **Local Area networks (LAN) :**



A



B



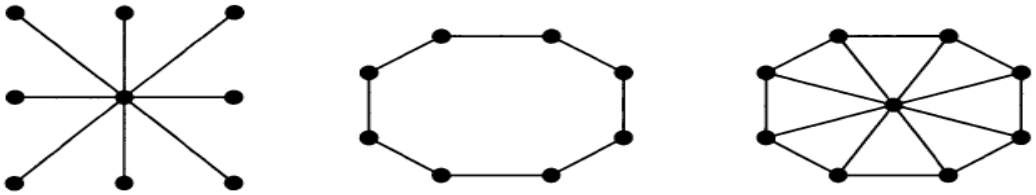
C

Various computer network can be connected using LAN .

1- LAN based on **star** topology as A each node connected to central device .

2- LAN based on a **ring** topology, as B ,each device connected exactly two others.

3.LAN use **hybrid** of A & B .message may be sent around the ring or through a central device as C (wheels) .



(a) (b) (c)

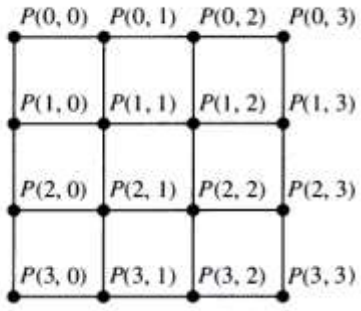
- **Parallel processing:**

We can use appropriate type of graph to represent the interconnection network of processors in computer.
the parallel algorithms break a problem into a number of sub problems that can be solved concurrently , them can be solved by using computer with multiple processors .
the simplest way to interconnect 4 processors to use an arrangement known as a linear array each processor connected to its neighbor except p(i-1), p(i+1).

<u>The mesh network</u> is used interconnection network.

The inside processors have 4 neighbours
The processors in corners have 2 neighbours
Other processors between corners have 3 neighbours

Linear array for 4 processors

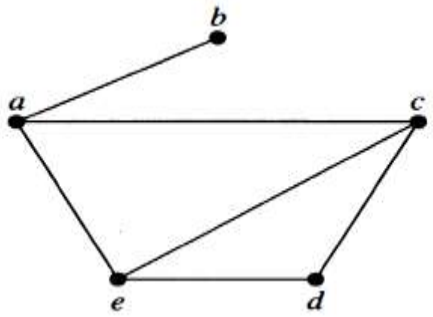


Mesh network for 16 processors

- **Representing Graphs :-**

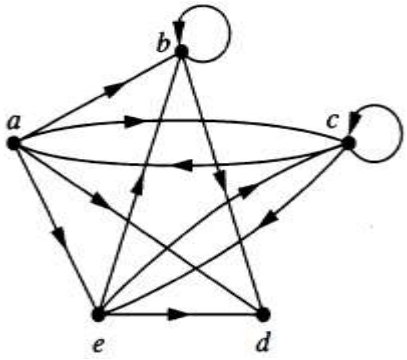
One way represent graphs without multiple edges is to list all the edges of the graph. Another way to represent a graph with no multiple edges is to use adjacency to each lists, which specify the vertices that are adjacent to each vertex of the graph.

1- adjacency list



An adjacency list for simple graph

a	b , c , e
b	a
c	a , d , e
d	c , e
e	a , c , d



An adjacency list for directed graph :-

a	b , c , d , e
b	b , d
c	a , c , e
d	---
e	c , d

2- A/ Adjacency Matrix

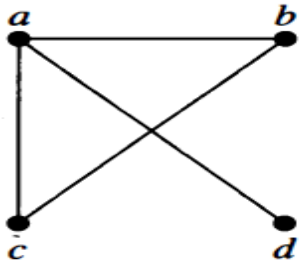
To simplify computation, graphs can be represent using matrices, 2 types of matrices used to represent graphs: one based on the adjacency of vertices, and the other is based on incidence of vertices and edges.

Suppose that G=(V,E) is simple graph, suppose the vertices of G are listed as $v_1 \ldots . v_n$ the adjacency matrix A is the n×n zero-one.

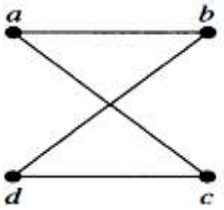
$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of G} \\ 0 & \text{otherwise} \end{cases}$$

The matrix represent the graph we order the vertices as a,b,c,d.

a	b	c	d
0	1	1	1
1	0	1	0
1	1	0	0
1	0	0	0



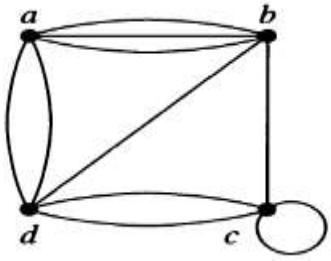
EX// Draw a graph with the adjacency matrix.



Adjacency Matrix

0	1	1	0
1	0	0	1
1	0	0	1
0	1	1	0

EX// Use an adjacency matrix to represent the graph :-



Adjacency Matrix

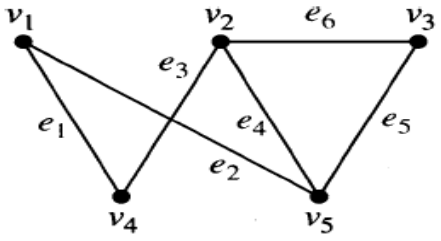
0	3	0	2
3	0	1	1
0	1	0	2
2	1	2	0

2-B/ Incidence Matrices

Another way to represent graphs is to use incidence matrices.
Let G=(V,E) be undirected graph (v_1 ,…., v_n) are vertices, (e_1 ,….., e_m) are edges.
The incidence matrix $[m_{ij}]$:-

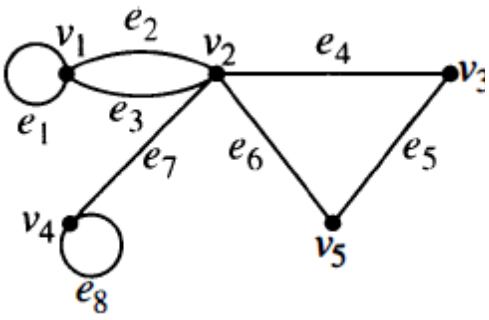
$$m_{ij}=\begin{cases} 1 & \text{when edge } e_j \text{ is incidence with } v_i \\ 0 & \text{otherwise} \end{cases}$$

EX1// Represent the graph with an incidence matrix :-



	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

EX 2// Represent the pseudograph using incidence matrix :-



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

- **Isomorphism of graphs**

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphism if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2.
Isomorphism of a simple graph is an equivalence relation.

Division// let G=(V,E) and G`=(V`, E`) be graphs, G and G` are said to be isomorphic if there exit pair of function :

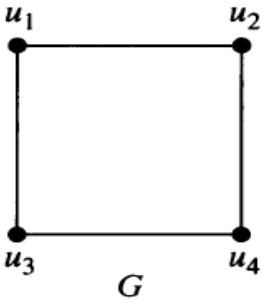
f : V ⟶ V` and g : E ⟶ E` such that

*f associates each element in V with exactly one element in V` and vice versa.
*g associates each element in E with exactly one element in E` and vice versa.
To prove 2 graphs are isomorphic :

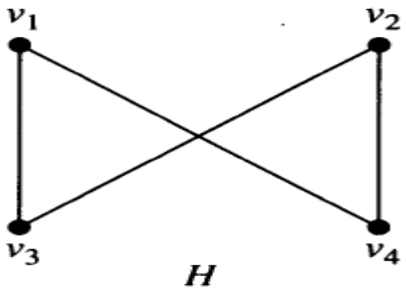
1) same number of vertices
2) same number of edges
3) same number of loops
4) same number of vertices of degree
5) same degree for corresponding vertices

EX// Consider the following graphs are they isomorphism (same) ?

1))



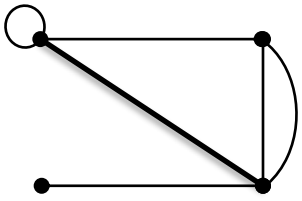
G



H

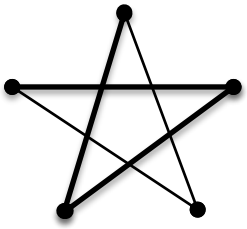
2))

G

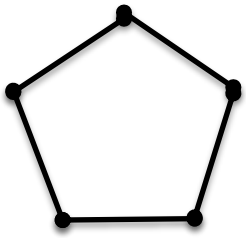


H

3))

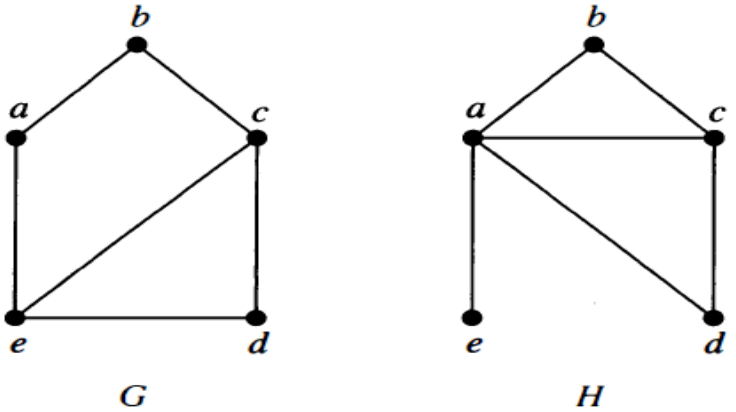


G



H

4))



Solution:

1) G , H have
   1. number of vertices= 4
   2. number of edges= 4
   3. number of loops= 0
   4. number of vertices of degree= 4 vertices of degree 2
   5. degree for corresponding vertices

   f(U1) = V1 , f(U2) = V4 , f(U3) = V3 , and f(U4) = V2 is a one-to-one correspondence between V and W.

   To see that this correspondence preserves adjacency, note that adjacent vertices in G are U I and U2 , U I and U 3 , U2 and U4, and U3 and U4, and each of the pairs f(U1) = V1 and f(U2) = V4, f(U1) = V I and f(U3) = V3 , f(U2) = V4 and f(U4) = V2 ,and f(U3) = V3 and f(U4) = V2 are adjacent in H .

∴ So we conclude that graphs G,H are they isomorphism

2)) G , H have
   1. number of vertices= 4
   2. number of edges= 4 and 5
   3. number of loops= 1

∴ So we conclude that graphs G,H are NOT isomorphism

3)) G , H have

1. number of vertices= 5
2. number of edges= 5
3. number of loops= 0
4. number of vertices of degree= 5 vertices of degree 2
5. degree for corresponding vertices

$G=(V,E)$

$V=\{V_1 , V_2 , V_3 , V_4 , V_5\}$

$E=\{(v_1,v_2) , (v_2,v_3) , \dots ,(v_4,v_5)\}$

$=\{ e_1 , e_2 , e_3 , e_4 , e_5 \}$

$G`=(V`, E`)$

$V`=\{V`_1 , V`_2 , V`_3 , V`_4 , V`_5\}$

$E`=\{(v`_1,v`_2) , (v`_2,v`_3) , \dots ,(v`_4,v`_5)\}$

$=\{ e`_1 , e`_2 , e`_3 , e`_4 , e`_5 \}$

Construct 2 functions

$f : V \longrightarrow V`$

$g : E \longrightarrow E`$

$f : V \longrightarrow V`$ $\quad$ $g : E \longrightarrow E`$

$V \longrightarrow V`$ $\quad$ $E \longrightarrow E`$

$V_1 \longrightarrow V`_1$ $\quad$ $E_1 \longrightarrow E`_1$

$V_2 \longrightarrow V`_2$ $\quad$ $E_2 \longrightarrow E`_2$

$V_3 \longrightarrow V`_3$ $\quad$ $E_3 \longrightarrow E`_3$

$V_4 \longrightarrow V`_4$ $\quad$ $E_4 \longrightarrow E`_4$

$V_5 \longrightarrow V`_5$ $\quad$ $E_5 \longrightarrow E`_5$

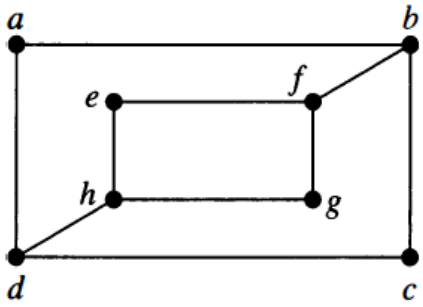
$\therefore$ So we conclude that graphs G,H are they isomorphism

4)) G , H have

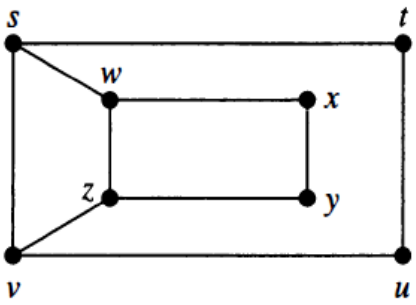
1. number of vertices= 5
2. number of edges= 6
3. number of loops= 0
4. number vertices of degree in G=3 vertices of degree 2, 2 vertices of degree 2 number of vertices of degree in H= 2 vertices of degree 2,2 vertices of degree, 1 vertices of degree 1, where H has a vertex of degree one, namely, e , whereas G has no vertices of degree one.

It follows that G and H are not isomorphic

EX// Determine whether the followings graphs are isomorphic.



G



H

Solution:

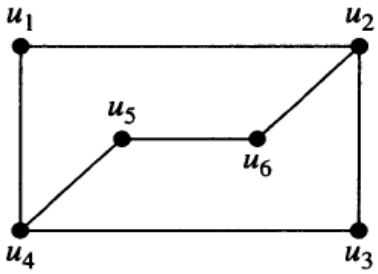
G , H have

1. number of vertices= 8
2. number of edges= 10
3. number of loops= 0
4. number of vertices of degree= 4 vertices of degree 2 and 4 vertices of degree 3
5. degree for corresponding vertices

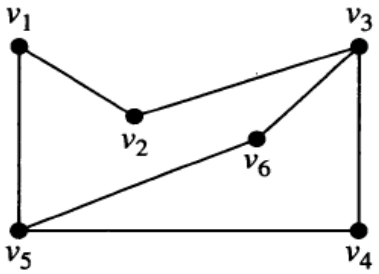
G and H are not isomorphic. To see this, note that because deg(a) = 2 in G, a must correspond to either t, u , x , or y in H, because these are the vertices of degree two in H . However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

∴ So we conclude that graphs G,H are NOT isomorphism

EX2// Determine whether the followings graphs are isomorphic.



G



H

Solution:

G , H have

1. number of vertices= 6
2. number of edges= 7
3. number of loops= 0
4. number of vertices of degree= 4 vertices of degree 2 and 4 vertices of degree 3
5. degree for corresponding vertices

$f(u_1) = v_6$ or v_4
$f(u_2) = v_3$ or v_5
$f(u_3) = v_4$
$f(u_4) = v_5$ or v_3
$f(u_5) = v_1$
$f(u_6) = v_2$

so $f(u_1) = v_6$
$f(u_2) = v_3$
$f(u_3) = v_4$
$f(u_4) = v_5$
$f(u_5) = v_1$
$f(u_6) = v_2$

$$\mathbf{A}_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array},$$

$$\mathbf{A}_H = \begin{array}{c} \\ v_6 \\ v_3 \\ v_4 \\ v_5 \\ v_1 \\ v_2 \end{array} \begin{array}{c} \begin{array}{cccccc} v_6 & v_3 & v_4 & v_5 & v_1 & v_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Because $A_G = A_H$, it follows that preserves edges, we conclude that f is isomorphism