-(Linear programming)

- Constructing Linear Programming Models

1-Forms of Linear programming (LP) model:

2- The formulation of linear programming Model

3-Method of solution of Linear programming Model

1. **Graphical method**

2. Simplex Method

4- Artificial Variable Technique
1. Two Phase Method
5- Duality in Linear Programming
-Duality and simplex method

6-The Transportation problem

-.Method for Initial Basic Feasible Solution to a transportation problems
1.North-West Corner Rule
2. Least Cost Method
3. Vogel's Approximation Method

7-Testing initial basic feasible solution and obtain by it the optimal solution

To find the optimal solution there are two methods.

1.Stepping Stone Method:

2. Modified Distribution method

8. Assignment problems

Operation Research

The first formal activities of operations research (OR) were initiated in England during second world war. When a team of British scientists set out to make scientifically based on decisions regarding the best utilization of war material. After the war the ideas advanced in military operation were adapted to improve efficiency and productivity in the civilian secter.

(Linear programming)

Linear programming is a deterministic mathematical technique, which involves the all location of scarce resources (machinery , Labor, Money, Time, Warehouse space, and raw material) in an optimal manner, based on a given criterion of optimality frequently, the criterion of optimality is either maximum profit or minimum cost, depending on the type of problem.

A linear programming (LP) model provides an efficient method for determining and optimal decision chosen from a large number of possible decisions. The optimal decision is the one that meet a specified objective of management subject to various restrictions and constraints.

Constructing Linear Programming Models

(Requirements to construct a linear programming models)

1. Objective Function: There must be an objective the firm wants to achieve, maximize profit or minimize cost.

2. Restriction and Decisions: There must be alternative courses of action or decisions, one of which will achieve the objective.

3. Linear objective function and linear constraints. We must be able to express the decision problem incorporating the objective and restrictions on the decisions using only linear equation and linear inequalities. That is, we must be able to state the problem as a linear programing model.

1-Forms of Linear programming (LP) model:

1.General form:-

The general linear programming form can be expressed as follows. Find the values of variables $X_1, \dots, X_n$ *which maximize (or minimize)an* objective function which is a linear function of variables, such as

$$[\ Max\ or\ Mim\ Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n\ ]$$

Subject to

$$\begin{cases} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1m}X_n \leq, = , \geq b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2m}X_n \leq, =, \geq b_2 \\ \vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n \leq, =, \geq b_m \end{cases}$$

And meet the non-negativity restrictions

$[X_1, X_2, \dots, X_n \geq 0]$.

Here $X_{j,}\ j = 1, \dots, n$ are called decision variables, $C_j, a_{ij,}\ and\ b_i, i = 1, \dots, m;\ j = 1, \dots, n$ are constants determined from the statement of the problem, the constants C_j represent the net unit contribution of decision variables X_j to the value of objective function and are called objective function coefficients, constants b_i, denote the total availability of the ith resource and are called stipulations and constants $a_{ij,}$ stand for the amount of resource, say i consumed per unit of variable (activity) X_j and are called structural coefficients.

2.Cononical Form:-

The general linear programming problem can always be put in the following form , called the canonical form:

$$[\ Max\ Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n\ ]$$

Subject to constraints

$$\begin{cases} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1m}X_n \leq b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2m}X_n \leq b_2 \\ \vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n \leq b_m \end{cases}$$

$[X_1, X_2, \dots, X_n \geq 0]$

The characteristics of this form are

(a) objective function is of maximization type,

(b) all constraints are of the $\leq type,$ (except non-negativity restrictions which are $\geq type$)

(c) all decisions variables are non-negative.

Remark :- Any linear programming problem can be put in the canonical form by use of some elementary transformations.

1. The minimization of a function, $f(x)$, is equivalent to the maximization of the negative expression of the function $-f(x)$, for example, the linear objective function

$$\min Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n$$

Is equivalent to,

$$Max\ G = -Z = -C_1X_1 - C_2X_2 - \cdots - C_nX_n \text{ with } Z = -G.$$

There for, for all linear programming problems the objective function can be expressed in the maximization form.

2. An inequality constraint of $(\geq)$ $type$ can be changed to an inequlity of $(\leq)$ by

Multiplying each side of the inequality by $(-,1)$ for example, the linear constraint

$$a_1X_1 + a_2X_2 \geq b, \ \ is\ equivalent\ to$$

$$-a_1X_1 - a_2X_2 \leq -b$$

3. An equation may be replaced by two weak inequalities in opposite direction. For example

$$a_1X_1 + a_2X_2 = b,$$

is equivalent to the tow simultaneous constraints

$$a_1X_1 + a_2X_2 \leq b \ \ and \ \ a_1X_1 + a_2X_2 \geq b \text{ or}$$

$$a_1X_1 + a_2X_2 \leq b \ \ and \ \ -a_1X_1 - a_2X_2 \leq -b$$

4. A inequality constraints with absolute form on the left hand side can be expressed as a combination of two regular inequalities. For example, for $b \geq 0$,

$|a_1X_1 + a_1X_2| \leq b$ is equivalent to

$a_1X_1 + a_2X_2 \leq b \;\; and \;\; a_1X_1 + a_2X_2 \geq -b$ or

$a_1X_1 + a_2X_2 \leq b \;\; and \;\; -a_1X_1 - a_2X_2 \leq b$

Similarly, for $q \geq 0$, $|p_1X_1 + p_2X_2| \geq q \quad is\ equivalent\ to$

$p_1X_1 + p_2X_2 \geq q \quad and \quad p_1X_1 + p_2X_2 \leq -q$ or

$-p_1X_1 - p_2X_2 \leq -q \quad and \quad p_1X_1 + p_2X_2 \leq -q$

5. If the decisions variables are unconstrained (unrestricted) in sign, that is, it may be positive, zero or negative. They can be expressed as the difference between two non-negative variables. For example, $X = X' - X''\ where\ X' \geq 0\ and\ X'' \geq 0$. Value of $X\ is\ positive\ , zero\ or\ negative\ depending\ on\ whether\ X'$is larger, equal or smaller than X''.

Remark: A minimization problem can also be in canonical form if all its variables are non-negative and all its constraints are of $(\geq)\ type$.

3.Standard Form

$$Max\ or\ Mim\ Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n$$

s.t:

$$\begin{cases} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1m}X_n = b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2m}X_n = b_2 \\ \vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \end{cases}$$

$[X_1, X_2, \dots, X_n \geq 0]$, $b_1, b_2, \dots, b_m \geq 0$

1.All variables are non-negative

2. The right hand side of each constraint is non-negative

3. All constraints are expressed as equations

4.Objective function may be of maximization or minimization type.

Remark: Any L.P. problem can be put in standard form with the help of some elementary transformation.

1.Any unrestricted variable X_i i, can be expressed as the difference of two non-negative variables

$i.e. X_i = X_i' - X_i''$, $X_i' \geq 0$, $X_i'' \geq 0$

2. If right hand side of a constraint is negative, it is multiplied on both sides by (-1) to make it positive. This will reverse the sign of inequality.

3. The inequality constraints are changed to equality constraints by adding or subtracting a non-negative variable from the left hand side of such constraints. These new variables are called slack variables. They are added if the constraints are (≤) and subtracted if the constraints are (≥). Since in the cost of (≥) constraints the subtracted variable represents the surplus of the left-hand side over right-hand side, it is commonly known as surplus variables and is, in fact a negative slack, Both decision variables as well as the slack and surplus variables are called the admissible variables. Slack variables are as much a part of the problem as decision variables and are treated in the same manner while finding a solution to the problem. These variables can remain positive throughout the process of solution and their values in the optimal solution give useful information about the problem. For example, the constraint

$\boldsymbol{a_1X_1 + a_2X_2 \leq b}$, $\mathbf{b \geq 0}$

is changed in in the standard form to

$\boldsymbol{X_1 + a_2X_2 + S_1 = b}$

where $S_1 \geq 0$. Also constraint $p_1X_1 + p_2X_2 \geq q$, q ≥ 0 is changed to

$p_1X_1 + p_2X_2 - S_2 = q, where\ S_2 \geq 0.$

The Quantities S_1 and S_2 are variables and their values depend on upon the values assumed by other $X's$ in a particular equation.

Example1: Change to canonical form

$Max\ Z = X_1 + 2X_3 - X_4$

S.t

$X_1 + X_2 + X_3 + X_4 = 10$

$X_2 + X_4 \geq 4$

$X_1 + X_3 \leq 8$

$|X_2 + X_3 - X_4| \leq 5$

$X_1, X_2 \geq 0$

Solution : Since the variables X_3, X_4 are not available in the non-negative restriction

Therefore they must be restricted in a signal as follow,

$Let\ \ X_3 = X_3' - X_3'',\ \ X_3', X_3'' \geq 0\ \ and\ \ \ X_4 = X_4' - X_4'',\ \ X_4', X_4'' \geq 0$

Substitute the values of X_3, X_4 $\ in\ the\ above\ restriction\ ,we\ get$

$Max\ Z = X_1 + 2(X_3' - X_3'') - (X_4' - X_4'')$

S.t

1. $X_1 + X_2 + (X_3' - X_3'') + (X_4' - X_4'') \leq 10$

   $-X_1 - X_2 - (X_3' - X_3'') - (X_4' - X_4'') \leq -10$

2. $-X_2 - (X_4' - X_4'') \leq -4$

3. $X_1 + (X_3' - X_3^{''))}) \leq 8$

4. $X_2 + (X_3' - X_3'') - (X_4' - X_4'') \leq 5$

$-X_2 - (X_3' - X_3'') + (X_4' - X_4'') \leq 5$

$X_1, X_2, X_3', X_3'', X_4', X_4'' \geq 0$

Homework: Express the following L.P. problems in to standard form:

(1) $Max\ Z = 7X_1 + 5X_2$

Subject to

$2X_1 + 3X_2 \leq 20$

$3X_1 + X_2 \geq 10$

$X_1, X_2 \geq 0$

2. $Max\ Z = 3X_1 + 2X_2 + 5X_3$

Subject to

$2X_1 - 3X_2 \leq 3$

$X_1 + 2X_2 + 3X_3 \geq 5$

$3X_1 + 2X_3 \leq 2$

$X_1, X_2 \geq 0$

3. $Max\ Z = 3X_1 + 5X_2 - 2X_3$

$X_1 + 2X_2 - X_3 \geq -4$

$-5X_1 + 6X_2 + 7X_3 \geq 5$

$2X_1 + X_2 + 3X_3 = 10$

$X_1, X_2 \geq 0\ , X_3\ is\ unrestricted\ in\ sign$

2-The formulation of linear programming Model

Example 2:

D and A company manufactures two products, each requiring a different manufacturing technique. The deluxe product requires 3 hours of labor, one hour of testing, and yields a profit of 10 dinars. The standard product requires 2 hours of labor, 2 hours of testing and yields profit 15 dinars. There are 15,000 hours of labor and 10,000 hours of testing available each year. A marketing forecast has shown that the yearly demand for the deluxe model to be no more that 7,000, and yearly demand for the standard model to be no more that 8,000 units. Management want to know the number of each model to produce yearly that will maximize total profit. Being this as linear programming problem.

Solution: $X_1 = the\ number\ of\ deluxe\ units$

$X_2 = the\ number\ od\ standard\ units$

	Deluxe	*Standard*	*total hours available yearly*
labor	3	2	15000
Testing	1	2	10000
Profit for each product	10	15	
Total demand	*7000*	8000	

Objective function: $Max\ Z = 10X_1 + 15X_2$

Constraints :

$3X_1 + 2X_2 \leq 15000$

$X_1 + 2X_2 \leq 10000$

$X_1 \leq 7000$

$X_2 \leq 8000$

$X_1, X_2, \geq 0$

Example 3: The standard weight of a special purpose brick is 5 kg and it contains two basic Ingredients $B_1, B_2. B_1.$ costs Rs. $5/kg$ and B_2 costs Rs. $8/kg$. Strength considerations dictate that the brick contains not more than 4 kg of $B_1 and\ a\ minimum\ of\ 2\ kg\ of\ B_2$.

Since the demand for the product is likely to be related to the price of the brick. Formulate this as linear programming model.

Solution: Let the quantities in kg of ingredients $B_1\ and\ B_2$ to be used to make the bricks be $X_1\ and\ X_2\ respectively$. Objective is to minimize the cost of the brick.

i.e., $Min\ Z = 5X_1 + 8X_2$

Constraints are

On the quantity of ingredient B_1 : $X_1 \leq 4$

On the quantity of ingredient B_2 : $X_2 \geq 2$

On the weight of the brick : $X_1 + X_2 = 5$

$X_1, X_2 \geq 0$

Example 4:The manufacturer of tires, the production of tires by mixing two types of rubber $A\ anb\ \ rubber\ B$. Each of them consists of four main components $\mu_1, \mu_2, \mu_3, and\ \mu_4$. As in the Following table

The basic component \ Rubber	Rubber A	Rubber B	Composition required
μ_1	—	0.45	1
μ_2	0.5	0.3	3
μ_3	0.35	—	1
μ_4	0.15	0.2	1.5
The cost per unit	32	24	

Write the formulation linear programming model

Solution: $Min\ Z = 32X_1 + 24X_2$

S.t

$0X_1 + 0.45X_2 \geq 1$

$0.5X_1 + 0.3X_2 \geq 3$

$0.35X_1 + 0X_2 \geq 1$

$0.15X_1 + 0.2X_2 \geq 1.5$

$X_1, X_2 \geq 0$

Homework: A Commercial company, three types of exported goods to the world market. As in the following table:

	Amount in the thousands of dinars			The amount allocated to the commodity
Expenditures	Commodity 1	Commodity 2	Commodity 3	
Marketing expenses	2	2	1	Equal 40000 Dinars
	2	1	2	At least 30000 Dinars
Administrative //various expenses	4	2	3	On the most equal 10000
Cost mode	5	4	3	

Write the formulation linear programming model.

3-Method of solution of Linear programming Model

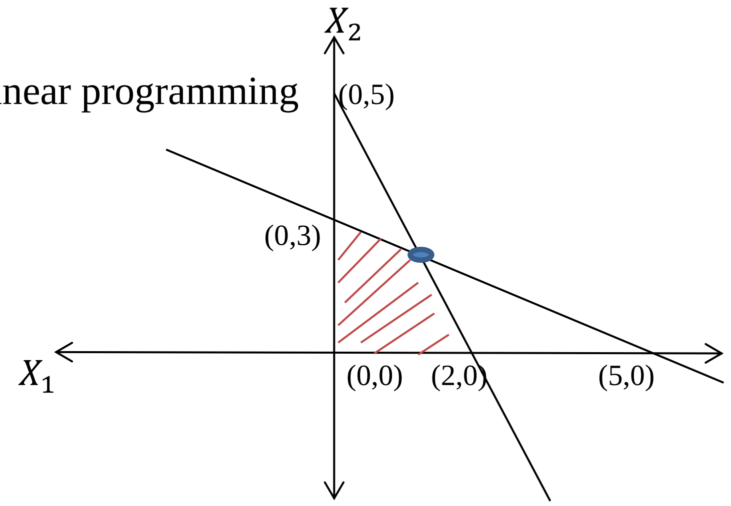
1. Graphical method

2. Simplex Method

not possible to solve linear programing problems graphically if the problem involving more than two decision variables. The graphical solution procedure required to develop a graph that displays the possible solutions, X_1 *and* X_2 values, where the values of X_1 is on the horizontal axis and value of X_2 on the vertical axis.

Any point on the graph can be identified by X_1 *and* X_2 values , so that every point on graph called a solution. The point $X_1 = 0$ and $X_2 = 0$ referred to the origin solution (initial solution). The solution we must consider is that point where the values of X_1 and $X_2 \geq 0$.

Example1: Find the optimal solution for linear programming problem



$Max\ z = 5X_1 + 3X_2$

S.t

$3X_1 + 5X_2 \leq 15$

$5X_1 + 2X_2 \leq 10$

$X_1, X_2 \geq 0$

Solution: Convert the first constraint into equation,

$3X_1 + 5X_2 \leq 15 \rightarrow 3X_1 + 5X_2 = 15$ and

Let $X_1 = 0 \rightarrow 3 \times (0) + 5X_2 = 15 \rightarrow X_2 = 3,\ \boldsymbol{A(0,3)}$

Let $X_2 = 0 \rightarrow 3X_1 + 5 \times (0) = 15 \rightarrow X_1 = 5,\ \boldsymbol{B(5,0)}$

Also Convert the second constraint into equation,

$5X_1 + 2X_2 \leq 10 \rightarrow 5X_1 + 2X_2 = 10$

$X_1 = 0 \rightarrow X_2 = 5, \qquad \boldsymbol{C(0,5)}$

$X_2 = 0 \rightarrow X_1 = 2\ ,\ \boldsymbol{D(2,0)}$

Since the points A and D is determined so we will determine the intersection point E

Now, we will find the intersection point E of the constraints by abbreviation method

$3X_1 + 5X_2 = 15 \qquad \times (5)$

$5X_1 + 2X_2 = 10 \qquad \times (3)$

$15X_1 + 25X_2 = 75$

$-15X_1 - 6X_2 = 30$

$19X_2 = 45 \;\rightarrow X_2 = 2.37,$ substitutes the value of X_2 in equation (1) to get the value $of\ X_1 which\ is\ X_1 = 1.05\ ,\ \boldsymbol{E(1.05, 2.37)}$

Points	$Max\ Z = 5X_1 + 3X_2$
A(0,3)	$5(0) + 3(3) = 9$
E(1.05,2.37)	$5(1.05) + 3(2.37) = 12.37$
(2,0)	$5(2) + 3(0) = 10$

The optimal solution of the given problem is $Max\ Z = 12.37\ at\ point\ (1.05, 2.37)$

Example 2: Find the optimal solution for linear programming problem.

$Max\ Z = 10X_1 + 2X_2$

S.t

$3X_1 + 2X_2 \leq 60$

$2X_1 + 4X_2 \leq 80$

$X_1, X_2 \geq 0$

Solution: Convert the first constraint into equation,

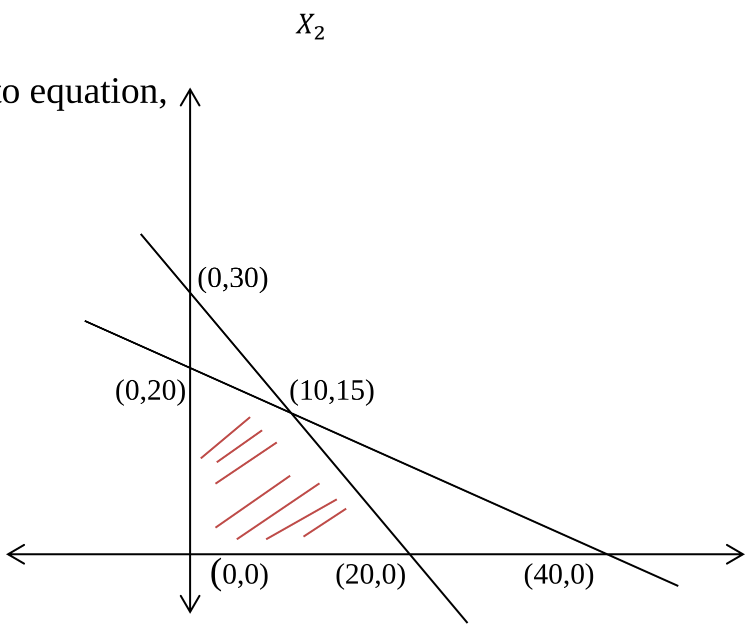
$3X_1 + 2X_2 = 60$

$let\ X_1 = 0 \rightarrow 2X_2 = 60 \;\rightarrow X_2 = 30$

Then the point $A(0,30)$

Let $X_2 = 0 \rightarrow 3X_1 = 60 \rightarrow X_1 = 20$

Then the point $B(20,0)$



Convert the Second constraint into equation,

$2X_1 + 4X_2 = 80$

Let $X_1 = 0 \rightarrow 4X_2 = 80 \rightarrow X_2 = 20\ , then\ the\ point\ C(0,20)$

Let $X_2 = 0 \rightarrow 2X_1 = 80 \rightarrow X_1 = 40\ , then\ the\ point\ D(40{,}0)$

Now, we will find the intersection point E of the constraints by abbreviation method

$3X_1 + 2X_2 = 60 \qquad \times (2)$

$2X_1 + 4X_2 = 80$,By subtraction second equation from first , we get

$6X_1 + 4X_2 = 120$

$-2X_1 - 4X_2 = 80$

$4X_1 =$
$40 \rightarrow\ X_1 = 10, substitute\ the\ value\ of\ X_1\ in\ equation\ (1) to\ get\ the\ value X_2$

$30 + 2X_2 = 60\ \rightarrow 2X_2 = 30 \rightarrow X_2 = 15\ , then\ the\ point\ E(10, 15)$

We choose the points B, C And E because they are common in the solution region.

	X_1	X_2	$Max\ Z = 10X_1 + 2X_2$
B	20	0	$10(20) + 12(0) = 200$
C	0	20	$10(0) + 12(20) = 240$
E	10	15	$\mathbf{10(10) + 12(15) = 280}$

The optimal solution of the given problem is 280 which is at point E(10,15)

Home work:

Find the optimal solution for the linear programming model

1. $Max\ Z = 4X_1 + 3X_2$

S.t

$5X_1 + 3X_2 \leq 30$

$2X_1 + 3X_2 \leq 21$

$X_1, X_2 \geq 0$

2. $Max\ Z = X_1 + 2X_2$

S.t

$X_1 + 2X_2 \leq 10$

$X_1 + X_2 \leq 1$

$X_2 \leq 4$

$X_1, X_2 \geq 0$

3. $Max\ Z = 3X_1 + 9X_2$

S.t

$X_1 + 4X_2 \leq 8$

$$X_1 + 2X_2 \leq 4$$
$$X_1, X_2 \geq 0$$

The Simplex Method (Technique or Algorithm)

The graphical method can not be applied when the number of variables involved in the L.P. problem is more than three or rather two. The simplex method can be used to solve any L.P. problem (for which the solution exists) involving any number of variables and constraints (hundred or even thousand).

The computational procedure in the simplex method is based on the fundamental property that the ***optimal solution*** to an L.P. problem, if it exists, occurs only at one of the corner points of the feasible region, which is one of the corner points of the feasible region. This solution is the tested i.e. it is ascertained whether improvement in the value of the objective function is possible by moving to the next corner point of the feasible region. If so the solution at this point is obtained. This search for better corner point is repeated, till after a finite number of trials, the optimal solution if it exists, is obtained.

For convenience, re-state The L.P. problem in standard form :

$$\boldsymbol{Max\ Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n + 0S_1 + 0S_2 + \cdots + 0S_m} \qquad \textbf{(1)}$$

Subject to constraints

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n + S_1 &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n + S_2 &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \qquad \vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n + S_m &= b_m \end{aligned} \qquad (2)$$

And $X_1, X_2, \dots, X_n, S_1, S_2, \dots, S_m \geq 0$ (3)

For easiness, an obvious starting basic feasible solution of m equations (2)is usually taken as:

$X_1 = X_2 = \cdots = X_n = 0\ ;\ S_1 = b_1,\ S_2 = b_2, \dots, S_m = b_m$. For this solution , the value of the objective function (1) is zero. $X_1, X_2, \dots, X_n$ $(each\ equal\ to\ zero)$ ***are non-basic variable*** and remaining non-zero variables $(S_1, S_2, \dots, S_m)$ $are\ basic\ variables.$

Example 1.Solve the L.P. problem

$$Max\ z = 3X_1 + 2X_2$$
$$subject\ to$$
$$X_1 + X_2 \leq 4$$
$$x_1 - x_2 \leq 2$$
$$X_1, X_2 \geq 0$$

Solution:

Step1: First, Observe whether all the right side constants of the constraints are non-negative. If not, it can be changed in to positive value on multiplying both sides of constraint by -1 . In this example all the b_i (right side constants) are already positive.

1- اولا نشاهد فيما اذا كان كل قيم الثوابت في الطرف الايمن غير سالبة . واذا وجد احد القيم سالب ، من الممكن تحويلة الى قيمة موجبة بضرب طرفي المتباينة بالعدد (-1) . في هذا المثال كل قيم الطرف الايمن موجبة .

Step 2: Express the problem into standard form.

Convert the inequality constraints to equations by introducing the non-negative slack or surplus variables. The coefficients of slack and surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '$\leq$' only slack variables S_1 *and* S_2 are needed. Therefore given problem now becomes:

$$Max\ z = 3X_1 + 2X_2 + 0S_1 + 0S_2$$
$$subject\ to$$
$$X_1 + X_2 + S_1 = 4$$
$$X_1 - X_2 + S_2 = 2$$
$$X_1, X_2, S_1, S_2 \geq 0$$

2- عبر عن المسالة بالشكل القياسي.
حول المتباينة الى معادلة بإضافة متغيرات مكملة او فائضة غير سالبة. دائما تؤخذ قيمة المعاملات للمتغيرات المكملة او الفائضة صفر في دالة الهدف. في هذا المثال كل قيود المتباينة تكون على الشكل $\leq$ لذلك فقط المتغيرات المكمل S_2, S_1 سوف نحتاج .

Step 3: Find the initial basic feasible solution and constraints a table **called complex table.**

We shall start with a basic solution which we shall get by assuming the profit earned is zero. This we will be so when non-basic variables $X_1 = X_2 = 0$.Column X_B gives the values of basic variables as indicated in the first column. Setting $X_1 = X_2 = 0$, the constraints yields the following ***initial basic feasible solution*** $S_1 = 4\ and\ S_2 = 2\ and\ Z = 0$. The above information can be expressed in the form of table (1) called ***simplex table***. Note: In this step, the variables $S_1\ and\ S_2$ are corresponding to the columns of basis matrix (identity matrix), so we will be called ***basic variables***. Other variables, $X_1\ and\ X_2$ are ***non-basic variables*** which always have the value zero**.**

3- سوف نبدء مع الحل الاساسي والذي نحصل عليه بفرض الربح يساوي صفر. وهذا يكون عندما تكون المتغيرات الغير الاساسية $X_1 = X_2 = 0$. يعطي العمود X_B قيم المتغيرات الاساسية كما مشار اليها في العمود الاول .
بوضع $X_1 = X_2 = 0$ تنتج القيود الحل المقبول الاساسي الابتدائي

$S_1 = 4\ and\ S_2 = 2\ and\ Z = 0.$

		C_j	3	2	0	0	
Basic variables	C_B	X_B	X_1	X_2	Basis Matrix S_1	S_2	Min ratio X_B/X_k for $X_k \geq 0$
S_1	$C_{B1} = (0)$	$X_{B1} = 4$	1	1	1	0	To be computed the next step
S_2	$C_{B2} = (0)$	$X_{B2} = 2$	1	-1	0	1	
	$Z = C_B X_B = 0$		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = Z_j - C_j = C_B X_j - C_j$

Remark:

1. The first row C_j, indicates the coefficients of the variables in the objective function. These coefficients remain unchanged in the subsequent table.

1.الصف الاول C_j يمثل معاملات المتغيرات في دالة الهدف . وان هذه المتغيرات تبقى قيمتها كما هي في الجداول اللاحقة

2. The second column (C_B) $column$ such that $C_B = (C_{B1}, C_{B2})$ in this example, represents The coefficients of the current basic variables in the objective function. For initial basic feasible solution, $C_B = (0,0)$ $in\ this\ example$.

2. العمود الثاني C_B يمثل معاملات المتغيرات الاساسية الحالية في دالة الهدف والمتمثلة في الجدول اعلاه . لذلك قيم تلك المعاملات الى الحل الاساسي الاولي المقبول هو $C_B = (0,0)$.

3. The first column is the basis column. It represents the basic variables of the current solution. In table (1) , the basic variables are the slack variables S_1, S_2.

3. العمود الاول هو العمود الاساس . الذي يمثل المتغيرات الاساسية في الحل الحالي . في الجدول (1) اعلاه ، المتغيرات الاساسية هي المتغيرات المكملة S_1, S_2.

4. The third column X_B $represents\ the\ value\ of\ the\ basic\ variables\ , that\ is,$ $X_B = (X_{B1}, X_{B2}) = (S_1, S_2)$. For initial basic feasible solution $X_B = (4, 2)$

4. يمثل العمود الثالث X_B قيم المتغيرات الاساسية ، وهذا يعني $X_B = (X_{B1}, X_{B2}) = (S_1, S_2)$. ويكون قيم تلك المتغيرات في الحل الاساسي الاولي المقبول هو $X_B = (4, 2)$.

Step 5: Proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the 'net evaluation' Δ_j $for\ each\ variable\ X_j$ by the formula.Thus, we get

$$\Delta_1 = C_B X_1 - C_1 = (0,0)(1,1) - 3 = (0 \times 1 + 0 \times 1) - 3 = -3$$
$$\Delta_2 = C_B X_2 - C_2 = (0,0)(1,-1) - 2 = (0 \times 1 - 0 \times 1) - 2 = -2$$
$$\Delta_3 = C_B S_1 - 0 = (0,0)(1,0) - 0 = (0 \times 1 + 0 \times 0) - 0 = 0$$
$$\Delta_4 = C_B S_2 - 0 = (0,0)(0,1) - 0 = (0 \times 0 + 0 \times 1) - 0 = 0$$

Remark:

Note that in the starting simplex table $\Delta_j's$ are same as $(-C_j)'s$. Also , $\Delta_j's$ corresponding to The columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test:

1.If all $\Delta_j (= Z_j - C_j) \geq 0$, the solution under test will be optimal. Alternative optimal solutions will exist if any non-basic Δ_j is also zero.

1. اذا كان كل قيم $\Delta_j (= Z_j - C_j) \geq 0$ سوف يكون الحل تحت الاختبار هو الحل الامثل .

2. If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

2. اذا كان واحد على الاقل من قيم Δ_j سوف يكون الحل تحت الاختبار حل غير امثل ، اذن تحسين الحل للوصول الى الحل الامثل بالخطوة التالية

3. If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero (≤ 0), then the solution under test will be ***unbounded.***

3. اذا كان كل قيم العمود X_j ***المناظرة الى اي قيمة سالبة*** Δ_j ***سالبة او صفر*** (≤ 0)***، فان الحل تحت الاختبار سوف يكون غير محدد.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 *and* Δ_2 both are negative. Hence we have to proceed to improve this solution in step 6.

بتطبيق هذه القواعد في اختبار امثلية الحل الى الحل الابتدائي الممكن ، سوف نشاهد Δ_1 *and* Δ_2 كلاهما سالب . لذلك سوف نستمر بتحسين الحل في الخوة القادمة .

Step6: In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector' and 'outgoing vector' respectively.***

الخطوة السادسة. لكي نحسن الحل الاساسي الممكن ، نحدد المتجه الداخل والمتجه الخارج من المصفوفة الاساس باستخدام القاعدة التالية .

'Incoming vector'. The incoming vector X_k is always selected corresponding to the most negative value of Δ_j **(sayΔ_k). Here** $\Delta_k = min(\Delta_1, \Delta_2) = min[-3, -2] = -3 = \Delta_1$ Therefore, $k = 1$ and hence column vector X_1 must enter the basis matrix . The column vector X_1 is marked by an upward arrow (↑).

المتغير الداخل. يختار دائما المتغير الداخل X_k المناظر الى القيمة السالبة الاكبر الى Δ_j **(sayΔ_k)** . وهنا **. Here** $\Delta_k = min(\Delta_1, \Delta_2) = min[-3, -2] = -3 = \Delta_1$ لذلك $k = 1$ والمتجه العمود X_1 يدخل المصفوفة الاساس . يكتب مع المتجه الداخل علامه الى الاعلى.

'outgoing vector'. The outgoing vector S_r is selected corresponding to minimum ratio of elements of X_B by the corresponding positive elements of predetermined incoming vector X_k. This rule is called the ***minimum ratio rule.*** This rule can be written as

المتجة الخارج. يختار المتجه الخارج S_r المناظر الى اقل نسبة من القيم الموجبة والمحددة من ناتج قسمة X_B على قيم المتجه X_1 .

$$\frac{X_{Br}}{X_{rk}} = \min[\frac{X_{Br}}{X_{ik}}, X_{ik} > 0]$$

$$For\ k = 1 \qquad \frac{X_{Br}}{X_{r1}} = min\left[\frac{X_{B1}}{X_{11}}, \frac{X_{B2}}{X_{21}}\right] = min\left[\frac{4}{1}, \frac{2}{1}\right]$$

$$\frac{X_{Br}}{X_{r1}} = \frac{2}{1} = \frac{X_{B2}}{X_{21}}$$

Comparing both sides of this equation, we get r=2. So that the vector S_2 marked with downward arrow (↓) should be removed from basis matrix. Now starting table (1) is modified to table (2).

بمقارنة كلا الجانبين من المعادلة ، نحصل على r=2 . لذلك المتجه S_2 والمؤشر بعلامة السهم (↓) يجب ان يزال من المصفوفة الاساس . الآن نكتب الجدول الثاني المعدل.

		C_j	3	2	0	0	
Basic variables	C_B	X_B	X_1	X_2	Basis Matrix S_1	S_2	Min ratio X_B/X_k $f0r$ $X_k \geq 0$
S_1	$C_{B1} = (0)$	$X_{B1} = 4$	1	1	1	0	$\frac{4}{1}$
$\boldsymbol{S_2}$	$C_{B2} = (0)$	$X_{B2} = 2$	↑ ← $[\mathbf{1}]$ →	-1	0	1	$\frac{\mathbf{2}}{\mathbf{1}}$ **min. ratio**
	$Z = C_B X_B = 0$		$\Delta_1 = \mathbf{-3}$ ↑ min. Δ_j Entering vector	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = Z_j - C_j = C_B X_j - C_j$

Step 7. In order to bring $S_2 = \begin{bmatrix}0\\1\end{bmatrix}$ in place of incoming vector $X_1 = \begin{bmatrix}1\\1\end{bmatrix}$, unity must occupy in the marked '[ ]' position and zero at all other places of X_1. If the number in the marked '[ ]' position is other than unity, divided all elements of that row by the 'key element'. (The element at the intersection of minimum ratio arrow (←) and incoming arrow (↑) is called the ***key element or pivot element*).**

الخطوة سبعة : لكي يحل المتغير الداخل X_1 محل المتغير الخارج S_2 يجب ان نحول key element والذي يحمل
العلامة '[ ]' الى **واحد** وجميع القيم في العمود الداخل X_1 الى اصفار ، وكذلك نقسم جميع القيم في ذلك الصف على key element . (العنصر والذي يقع عند تقاطع اقل خارج قسمة والمتغير الداخل يدعى ***key element*** (***or pivot element***

Then, subtract appropriate multiplies of this new row from the other (remaining)rows, so as to obtain zeros in the remaining positions of the column X_1. Thus, the process can be fortified by simplex matrix transformation as follows:
The intermediate coefficient matrix is :

بعد ذلك ، بضرب ذلك الصف بعدد مناسب وطرحه من الصفوف الاخرى لكي نحول بقية القيم في العمود X_1 الى اصفار

	X_B	X_1	X_2	S_1	S_2	
R_1	4	1	1	1	0	
R_2	2	1	-1	0	1	
R_3	Z=0	-3	-2	0	0	$\leftarrow \Delta_j$

Applying $R_1 \to R_1 - R_2,\ R_3 \to R_3 + 3R_2$ to obtain

X_B	X_1	X_2	S_1	S_2	
2	0	2	1	-1	
2	1	-1	0	1	
Z=6	0	-5	0	3	$\leftarrow \Delta_j$

Now, construct the improved simplex table (3) as follows:

		C_j	3	2	0	0	
Basic variables	C_B	X_B	X_1	$\boldsymbol{X_2}$	Basis Matrix S_1	S_2	Min ratio X_B/X_k for $X_k \geq 0$
$\boldsymbol{S_1}$	0	2	0	$[\mathbf{2}]$	1	-1	$\frac{2}{2} \leftarrow key\ row$
X_1	3	2	1	-1	0	1	(Negative value is not considered)
	$Z = C_B X_B = 6$		0	$\mathbf{-5}$ $\uparrow$	$\mathbf{0}$ $\downarrow$	3	$\Delta_j = Z_j - C_j = C_B X_j - C_j$

From this table, the improved basic feasible solution is read as : $X_1 = 2, X_2 = 0$, $S_1 = 2, S_2 = 0$ the improved value of $Z = 6$.

Remark: Note that $\Delta_j's$ are also computed while transforming the table by matrix method. However, the correctness of $\Delta_j's$ can be verified by computing them independently by using the formula

$$\Delta_j = Z_j - C_j = C_B X_j - C_j$$

Step 8: Now repeat step 5 through 7 as and when needed until an optimal solution is obtained table 3. $\Delta_k = most\ negative\ \Delta_j = \Delta_2$.

Therefore, $k = 2$ and hence X_2 should be the entering vector (key column). By minimum ratio rule : Minimum ratio$\left(\frac{X_B}{X_2}, X_2 > 0\right) = min\left[\frac{2}{2}, -\right]$ (since negative ratio is not counted, so the second ratio is not considered).

Since first ratio is minimum, remove the first vector S_1 from the basis matrix. Hence the key element is. Divided the first row by key element 2, the intermediate coefficient matrix is obtained as:

	X_B	X_1	X_2	S_1	S_2	
R_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	
R_2	2	1	-1	0	1	
R_3	Z=6	0	-5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \rightarrow R_2 + R_1, \quad R_3 \rightarrow R_3 + 5R_1$

X_B	X_1	X_2	S_1	15	
1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	
3	1	0	$\frac{1}{2}$	$\frac{1}{2}$	
Z=11	0	0	$\frac{5}{2}$	$\frac{1}{2}$	$\leftarrow \Delta_j$

Now, construct the improved simplex table (4) which is final simplex table as follows:

		C_j	3	2	0	0	
Basic variables	C_B	X_B	X_1	$\boldsymbol{X_2}$	**Basis Matrix** S_1	S_2	Min ratio X_B/X_k $f0r$ $X_k \geq 0$
X_2	2	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{2} \leftarrow key\ row$
X_1	3	3	1	0	$\frac{1}{2}$	$\frac{1}{2}$	(Negative value is not considered)
	$Z = C_B X_B = 11$		0	**0**	$\boldsymbol{\frac{5}{2}}$	$\frac{1}{2}$	$\Delta_j = Z_j - C_j = C_B X_j - C_j$

The solution as read from this table is $X_1 = 3, X_2 = 1, S_1 = 0, S_2 = 0, and\ Max\ Z = 11$. Also, using formula $\Delta_j = Z_j - C_j = C_B X_j - C_j$ verify that all $\Delta_j' s$ are non-negative Hence the optimal solution is $X_1 = 3, X_2 = 1, Max\ Z = 11$.

-Simple Way For Simplex Method Computations

Complete solution with its different computational steps can be more conveniently represented by the following single table.

		C_j	3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B/X_k
S_1	0	4	1	1	1	0	$\frac{4}{1}$
$\leftarrow S_2$	0	2	[1]	-1	0	1	$\frac{\mathbf{2}}{\mathbf{1}} \leftarrow \boldsymbol{Min}$
$X_1 = X_2 = 0$	$Z = C_B X_B = 0$		$\mathbf{-3^*}$ ↑	-2	0	0 ↓	$\Delta_j = Z_j - C_j = C_B X_j - C_j$
$\leftarrow S_1$	0	2	0	[2]	1	-1	$\frac{\mathbf{2}}{\mathbf{2}}$ **←Min**
X_1	3	2	1	-1	0	1	$-$
$X_2 = S_2 = 0$	$Z = C_B X_B = 6$		0	$\mathbf{-5^*}$ ↑	0	3	Δ_j
X_2	2	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	
X_1	3	3	1	0	$\frac{1}{2}$	$\frac{1}{2}$	
$S_1 = S_2 = 0$	$Z = C_B X_B = 11$		0	0	$\frac{5}{2}$	$\frac{1}{2}$	All$\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $\boldsymbol{X_1 = 3, X_2 = 1}$, $\boldsymbol{Max\ Z = 11}$.

Example 1.Solve the L.P. problem

$$Min\ z = X_1 - 3x_2 + 2X_3$$
$$subject\ to$$
$$3X_1 - X_2 + 3X_3 \leq 7$$
$$-2X_1 + 4X_2 \leq 12$$
$$-4x_1 + 3X_2 + 8X_3 \leq 10$$
$$X_1, X_2, X_3 \geq 0$$

Solution: This is the problem of minimization. Converting the objective function from minimization to maximization, we have

$$Max - Z = -X_1 + 3x_2 - 2X_3 + 0S_1 + 0S_2 + 0S_3 = Max\ Z' \ \ where - Z = Z'$$

Subject to $\quad 3X_1 - X_2 + 3X_3 + S_1 = 7$

$$-2X_1 + 4X_2 + 0X_3 + S_2 = 12$$
$$-4x_1 + 3X_2 + 8X_3 + S_3 = 10$$
$$X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$$

Setting $X_1 = X_2 = X_3 = 0$, the constraints yields the following ***initial basic feasible solution*** $S_1 = 7,\ S_2 = 12,\ S_3 = 10\ \ and\ Z = 0.$

$Here\ we\ give\ only\ table\ of\ solution.$

		C_j	-1	3	-2	0	0	0	
Basic variable	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	15
S_1	0	7	3	-1	3	1	0	0	$-$
S_2	0	12	-2	$[4]$	0	0	1	0	$12/4 = 3 \leftarrow Min$
S_3	0	10	-4	3	8	0	0	1	$10/3$
$X_1 = X_2 = X_3 = 0$	$Z = 0\ , Z' = 0$		1	$-3* \uparrow$	2	0	$0 \downarrow$	0	$\leftarrow \Delta_j$
S_1	0	10	$\left[\frac{5}{2}\right]$	0	3	1	$\frac{1}{4}$	0	$\frac{10}{\frac{5}{2}} = 4$
X_2	3	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	$-$
S_3	0	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	$-$
$X_1 = S_2 = X_3 = 0$	$Z' = 9\ , \therefore Z = -9$		$\left(-\frac{1}{2}\right) * \uparrow$	0	2	$0 \downarrow$	$\frac{3}{4}$	0	$\leftarrow \Delta_j$
X_1	-1	4	1	0	$\frac{6}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	
X_2	3	5	0	1	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	
S_3	0	11	0	0	11	1	$-\frac{1}{2}$	1	
$S_1 = S_2 = X_3 = 0$	$Z' = 11, \therefore Z = -11$		0	0	$\frac{13}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0	$\leftarrow \Delta_j \geq 0$

The optimal solution is: $X_1 = 4, X_2 = 5\ , X_3 = 0\ , Min\ Z = -11$

Homework:

1.Solve the L.P. problem

$$Max\ Z = 3X_1 + 2x_2 + 5X_3$$
$$subject\ to$$
$$X_1 + 2X_2 + X_3 \leq 430$$
$$3X_1 + 2X_3 \leq 460$$
$$x_1 + 4X_2 \leq 420$$
$$X_1, X_2, X_3 \geq 0$$

2.

$$Max\ Z = 10X_1 + 12x_2$$
$$subject\ to$$
$$2X_1 + 3X_2 \leq 15$$
$$3X_1 + 2X_2 \leq 16$$
$$x_1 + X_2 \leq 6$$
$$X_1, X_2 \geq 0$$

4-Artificial Variable Technique

1. Two Phase Method

Linear programming problems, in which constraints may also have $' \geq '$ and '$= '$ signs after ensuring that all $\boldsymbol{b_i}$ are $\geq \mathbf{0}$, are considered in this section . In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, ***the artificial variable***. These variables fictitious and can not have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table when they become zero (non-basic). The process of eliminating artificial variables is performed in ***phase 1*** of the solution, and ***phase 2*** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases, it is called ' two phase simplex method.

Remarks.

1. The objective of ***phase 1*** is to search for a basic feasible solution to the given problem it ends up either giving a basic feasible solution or indicating that the given L.P. problem has no feasible solution at all.

2. The basic feasible solution at the end of ***phase 1*** provides a starting basic feasible solution for the given L.P. problem. ***Phase 2*** is then just the application of simplex method to move towards optimality.
3. In ***phase 2***, care must be taken to ensure that an artificial variable is never allowed to become positive, if were present in the basis. Moreover, some artificial variable happens to leave the basis, its column must be deleted from the simplex table altogether.

5.2-2. Alternative Approach of Two-Phase simplex method

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase 1. In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.
Step 1. Assign a cost $\mathbf{-1}$ to each artificial variable and a cost $\mathbf{0}$ to all other variables (in place of their original cost) in the objective function.
Step 2. Construct the auxiliary linear programming problem in which the new objective function Z' is to be maximized subject to the given set of constraints.
Step 3. Solve the auxiliary problem by simplex method until either of following three possibilities do arise :
(a)$Max\ Z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.
(b) $Max\ Z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to **phase 2**.
(c) $Max\ Z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to ***phase 2***.
Phase 2: Now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appear in the basis at the zero level. This new objective function is now maximized by simplex method subject to given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of ***phase 1***, until an optimum basic feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of **phase** *1* are removed.

Example 1. Use two phase method to solve the problem:

$$Min\ Z = X_1 - 2X_2 - 3X_3$$

Subject to the constraints

$$-2X_1 + X_2 + 3X_3 = 2$$
$$2X_1 + 3X_2 + 4X_3 = 1$$
$$X_1, X_2, X_3 \geq 0$$

Solution. First convert the objective function into maximization form:
$Max\ Z' = -X_1 + 2X_2 + 3X_3$, Where $Z' = -Z$
Introducing the artificial variables $A_1 \geq 0\ and\ A_2 \geq 0$, the constraints of the given problem become,

$$-2X_1 + X_2 + 3X_3 + A_1 = 2$$
$$2X_1 + 3X_2 + 4X_3 + A_2 = 1$$
$$X_1, X_2, X_3, A_1, A_2 \geq 0$$

Phase 1. Auxiliary L.P. problem is

$$M\ ax\ Z'^* = -0X_1 + 0X_2 + 0X_3 - 1A_1 - 1A_2$$

Subject to the constraints

$$-2X_1 + X_2 + 3X_3 + A_1 = 2$$
$$2X_1 + 3X_2 + 4X_3 + A_2 = 1$$
$$X_1, X_2, X_3, A_1, A_2 \geq 0$$

Let $X_1 = X_2 = X_3 = 0$, *then the* initial basic feasible solution is $A_1 = 2, A_2 = 1$
The following solution table is obtained for auxiliary problem

		C_j	0	0	0	-1	-1	
Basic Variable	C_B	X_B	X_1	X_2	X_3	A_1	A_2	Min. Ratio X_B/X_k
A_1	-1	2	-2	1	3	1	0	$2/3$
A_2	-1	1	2	3	[**4**]	0	1	$1/4 \leftarrow$
$X_1 = X_2 = X_3 = 0$	$Z'^* = -3$		0	-4	-7^* ↑	0	0	$\leftarrow \Delta_j$
A_1	-1	$5/4$	$-7/2$	$-5/4$	0	1	$-3/4$	
X_3	0	$1/4$	$1/2$	$3/4$	1	0	$1/4$	
	$Z'^* = -5/4$		$7/2$	$5/4$	0	0	$7/4$	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution to the auxiliary L.P.P. has been attained. But at the same time $Max\ Z'^*$ is negative and the artificial variable A_1 appears in the basic solution at a positive level. Hence the original problem does not possess any feasible solution. Here there is no need to enter Phase 2.

Example 1. Use two phase method to solve the problem:

$$Min\ Z = \frac{15}{2}X_1 - 3X_2$$

Subject to the constraints

$$3X_1 - X_2 - X_3 \geq 3$$
$$X_1 - X_2 + X_3 \geq 2$$
$$X_1, X_2, X_3 \geq 0$$

Solution. Phase one. Convert the objective function into maximization form:

$$Max\ Z' = -\frac{15}{2}X_1 + 3X_2.$$

Introducing the surplus variables $S_1 \geq 0,\ S_2 \geq 0$ *and artificial variables* $A_1 \geq 0, A_2 \geq 0$, the constraints of the given problem become

$$3X_1 - X_2 - X_3 - S_1 + A_1 = 3$$
$$X_1 - X_2 + X_3 - S_2 + A_2 = 2$$
$$X_1, X_2, X_3, S_1, S_2, A_1, A_2 \geq 0$$

Phase 1. Assigning a cost -1 to artificial variables A_1 *and* A_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes:

$$Max\ Z'^{*} = 0X_1 + 0X_2 + 0X_3 + 0S_1 + 0S_2 - 1A_1 - 1A_2$$

Subject to the constraints

$$3X_1 - X_2 - X_3 - S_1 + A_1 = 3$$
$$X_1 - X_2 + X_3 - S_2 + A_2 = 2$$
$$X_1, X_2, X_3, S_1, S_2, A_1, A_2 \geq 0$$

Let $X_1 = X_2 = X_3 = S_1 = S_2 = 0$, *then the* initial basic feasible solution is $A_1 = 3, A_2 = 2$. Now apply simplex method in usual manner,

		C_j	0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	A_1	A_2	Min. Ratio X_B/X_k
A_1	-1	3	**[3]**	-1	-1	-1	0	1	0	$3/3$ ←
A_2	-1	2	1	-1	1	0	-1	0	1	$2/1$
	$Z'^{*} = -5$		-4^{*} ↑	2	0	1	1	0	0	← Δ_j
X_1	0	1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	–
A_2	-1	1	0	$-2/3$	$\left[\mathbf{4/3}\right]$	$1/3$	-1	$-1/3$	1	$3/4$
	$Z'^{*} = -1$		0	$2/3$	$-4/3^{*}$ ↑	$-1/3$	1	$4/3$	0	← Δ_j
X_1	0	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	
X_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	
	$Z'^{*} = 0$		0	0	0	0	0	1	1	← $\Delta_j \geq 0$

Since All $\Delta_j \geq 0$ and no artificial variable appears in the basic, an optimal solution to the auxiliary problem has been attained.

Phase 2. In this phase now consider the actual costs associated with original variables, the objective function thus becomes $Max\ Z' == -\frac{15}{2}X_1 + 3X_2 + 0S_1 + 0S_2$.

Now apply simplex method in the usual manner.

		C_j	$-\frac{15}{2}$	3	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min. Ratio X_B/X_k
X_1	$-15/2$	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	
X_3	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	
	$Z' = -75/8$		0	$3/4$	0	$15/8$	$15/8$	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution has been attained.

Hence optimal solution is : $X_1 = 5/4,\ X_2 = 0,\ X_3 = 3/4\ ,\ Min\ Z = 75/8$

Homework. Solve the problem by two phases method **:**

(1)

$$Min\ Z = X_1 + X_2$$

Subject to the constraints

$$2X_1 + X_2 \geq 4$$
$$X_1 + 7X_2 \geq 7$$
$$X_1, X_2 \geq 0$$

(2)

$$Max\ \ Z = 5X_1 + 8X_2$$

Subject to the constraints

$$3X_1 + 2X_2 \geq 3$$
$$X_1 + 4X_2 \geq 4$$
$$X_1 + X_2 \leq 5$$
$$X_1, X_2 \geq 0$$

(3)

$$Max\ \ Z = 3X_1 - X_2$$

Subject to the constraints

$$2X_1 + X_2 \geq 2$$
$$X_1 + 3X_2 \leq 4$$
$$X_2 \leq 4$$
$$X_1, X_2$$

5- Duality in Linear Programming

For every L.P. problem there is a related unique L.P. problem (another linear programming).

The given original programme is called the primal programme (P). This programme can be rewritten by transposing (reversing) the rows and columns of the algebraic statement of the problem. Inverting the pergramme in this way results in ***dual programme*** (D). The variables of dual programme are known as dual variables or shadow prices of the various resources. A solution to the dual programme may be found in a manner similar to that used for the primal. The two programmes have very closely related properties so that ***optimal solution*** of the dual problem gives complete information about the ***optimal solution*** of the primal problem and vice versa.

Duality is an extremely important and interesting feature of linear programming. The various aspects of this property are

(1) If the primal problem contains a large number of rows (constraints) and smaller number of columns (variables), the computational procedure can be considerably reduced by converting it into dual and then solving it. Hence it offers an advantage in many applications.

(2) It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of the problem. This form the basis of post optimality or sensitivity analysis.

(3) Duality in linear programming has certain far reaching consequence of economic nature.

This can help managers answer questions about alternative courses of action and their relative values.

(4) Calculation of the dual checks the accuracy of the primal solution.

(5) Duality in linear programming shows that each linear programme is equivalent to a two person zero-sum game. This indicate that fairly close relationships exist between linear programming and the theory of games.

(6) Duality is not restricted to linear programming problems only but finds application in economics, physics and other fields. In economics it is used in the formulation of input and output systems. In physics it is used in the series circuit and parallel circuit theory.

(7) Economics interpretation of the dual helps the management in making future decisions.

(8) Duality is used to solve L.P. problems(by the dual simplex method) in which the initial solution is infeasible.

(9) The solution of the dual problem can be used by the decision-maker for planning or augmenting (increasing) the resources.

1. Dual problem when primal is in canonical form

The general programming problem in canonical form as discussed before:

$$Max\ Z = C_1X_1 + C_2X_2 + C_3X_3 \dots + C_nX_n$$

S.t.

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \dots + a_{1n}X_n &\le b_1 \\ a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \dots + a_{2n}X_n &\le b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + \dots + a_{mn}X_n &\le b_m \end{aligned} \qquad (1)$$

Where

$$X_1, X_2, X_3, \dots, X_n \ge 0$$

If the above problem is referred to as primal, then its associated dual will be

$$Min\ W = b_1Y_1 + b_2Y_2 + b_3Y_3 + \cdots + b_mY_m$$

S.t.

$$\begin{aligned} a_{11}Y_1 + a_{21}Y_2 + a_{31}Y_3 + \dots + a_{m1}Y_m &\ge C_1 \\ a_{12}Y_1 + a_{22}Y_2 + a_{32}Y_3 + \dots + a_{m2}Y_m &\ge C_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\ a_{1n}Y_1 + a_{2n}Y_2 + a_{3n}Y_3 + \dots + a_{mn}Y_m &\ge C_n \end{aligned} \qquad (2)$$

Where the dual variables $Y_1, Y_2, Y_3, \dots, Y_m \ge 0$

Equations (1) and (2) are called symmetric primal –dual pairs.

The above pair of programs can be written as

Primal	**Dual**
$Maz\ Z = \sum_{j=1}^{n} C_jX_j$	$Min\ W = \sum_{i=1}^{m} b_iY_i$
Subject to	Subject to
$\sum_{j=1}^{n} a_{ij}X_j \le b_i$, i=1,2,3, …,$m$,	$\sum_{i=1}^{m} a_{ij}Y_i \ge C_j$,
$j = 1,2,3, \dots, n$,	
Where $X_j \ge 0, j = 1,2,3, \dots, n$.	where $Y_i \ge 0$,
	$i = 1,2,3, \dots, m$.

From above two programs, the following points are clear:

(1)If the primal contains n variables and m constraints, the dual will contains m variables and n constraints.

(2) the maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(3) The maximization problem has ($\le$) constraints while the minimization problem has ($\ge$) constraints.

(4) Constraints of ($\leq$) type in the primal becomes ($\geq$) $type\ in\ the\ dual\ and$ vce versa.
(5) The coefficient matrix of the constraints of the dual is the transpose of the primal.
(6) A new set of variables appear in the dual.
(7) The constants $C_1, C_2, C_3, \dots, C_n$ in the objective function of the primal appear in the constraints of the dual.
(8) The constants $b_1, b_2, b_3, \dots, b_m$ in the constraints of the primal appear in the objective function of the dual.
(9) The variables in the both problems are non-negative.
The constraints relationships of the primal and dual can be represented in a single table as follows :

	X_1	X_2	X_3	$\cdots$	X_n	
Y_1	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leq b_1$
Y_2	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leq b_2$
Y_3	a_{31}	a_{32}	a_{33}	$\cdots$	a_{3n}	$\leq b_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Y_m	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leq b_m$
	$\geq C_1$	$\geq C_2$	$\geq C_3$	$\vdots$	$\geq C_n$	

Example 1: Construct the dual to the primal problem

$$Max\ Z = 3X_1 + 5X_2$$

S.t.

$$2X_1 + 5X_2 \leq 50$$
$$3X_1 + 2X_2 \leq 35$$
$$5X_1 - 3X_2 \leq 10$$
$$X_2 \leq 20$$

Where $X_1, X_2 \geq 0$

Solution: Let Y_1 , Y_2 Y_3 and Y_4 be the corresponding dual variables, then the dual problem is given by

$$Man\ W = 50Y_1 + 35Y_2 + 10Y_3 + 20Y_4$$

S.t.

$$2Y_1 + 3Y_2 + 5Y_3 \geq 3$$
$$5Y_1 + 2Y_2 - 3Y_3 + Y_4 \geq 5$$

Where $Y_1, Y_2, Y_3, Y_4 \geq 0$

Remark: As the dual problem has the lesser number of constraints than the primal (2 instead of 4), it requires lesser work and effort to solve it. This follows from the fact that the computational difficulty in the linear programming problem is mainly associated with the number of constraints rather than number of variables.

Example 2: construct the dual of the problem

$$Min\ Z = 3X_1 - 2X_2 + 4X_3$$

Subject to the constraint

$$3X_1 + 5X_2 + 4X_3 \geq 7$$
$$6X_1 + X_2 + 3X_3 \geq 4$$
$$7X_1 - 2X_2 - X_3 \leq 10$$
$$X_1 - 2X_2 + 5X_3 \geq 3$$
$$4X_1 + 7X_2 - 2X_3 \geq 2$$
$$X_1 , X_2 , X_3 \geq 0$$

Solution : As the given problem of minimization, all constraints should be of $\geq type$. Multiplying the third constraints by -1 on both sides, we get

$$-7X_1 + 2X_2 + X_3 \geq -10.$$

The dual of given problem will be

$$Max\ W = 7Y_1 + 4Y_2 - 10Y_3 + 3Y_4 + 2Y_5$$

Subject to

$$3Y_1 + 6Y_2 - 7Y_3 + Y_4 + 4Y_5 \leq 3$$
$$5Y_1 + Y_2 + 2Y_3 - 2Y_4 + 7Y_5 \leq -2$$
$$4Y_1 + 3Y_2 + Y_3 + 5Y_4 - 2Y_5 \leq 4$$
$$Y_1 , Y_2 , Y_3 , Y_4 , Y_5 \geq 0$$

Where $Y_1 , Y_2 , Y_3 , Y_4\ and\ Y_5$ are the dual variables associated with the first, second, third, fourth and fifth constraint respectively.

Homework: construct the dual of the problem

(1)

$$Max\ Z = 3X_1 + 17X_2 + 9X_3$$

Subject to constraints

$$X_1 - X_2 + X_3 \geq 3$$
$$-3X_1 + 2X_3 \leq 1$$
$$X_1, X_2, X_3 \geq 0$$

(2)

$$Max\ Z = X_1 + X_2 - X_3 - X_4$$

Subject to constraints

$$3X_1 - 2X_2 + X_3 + 5X_4 \leq 18$$
$$5X_1 + 6X_3 \leq 20$$
$$X_1 - X_2 + 4X_3 + X_4 \geq 9$$
$$X_1, X_2, X_3, X_4 \geq 0$$

(3)

$$Min\ Z = 2X_1 + X_2$$

Subject to constraints

$$3X_1 + X_2 \geq 3$$

$$4X_1 + 3X_2 \geq 6$$
$$X_1 + 2X_2 \leq 3$$
$$X_1, X_2 \geq 0$$

(4)

$$Max\ Z = 2X_1 - X_2$$
Subject to constraints
$$X_1 + 3X_2 = 7$$
$$X_1 - X_2 = 3$$
$$X_1, X_2 \geq 0$$

2.Duality and simplex method

(The final simplex method table giving optimal solution of the primal also contains optimal solution of its dual in itself, and conversely. This is based on the 'Fundamental Duality theorem' which is related as follows:

(a) If either the primal or the dual problem has a finite optimal solution, then the other problem also has a finite optimal solution. Furthermore, the optimal values of the objective function in the both the problems are the same, this mean that, $MaxZ = Min\ W$.

(b) If either problem has an unbounded optimal solution, then the other problem has no feasible solution at all.

(c) Both problem may be infeasible.

5-2-1. Comparison of solutions to the primal and its dual

Example 3: Consider the following pair of dual problem :

$Max\ Z = 40X_1 + 50X_2$
Subject to
$2X_1 + 3X_2 \leq 3$

$8X_1 + 4X_2 \leq 5$
and $X_1, X_2 \geq 0$

$Min\ W = 3Y_1 + 5Y_2$
subject to
$2Y_1 + 8Y_2 \geq 40$

$3Y_1 + 4Y_2 \geq 50$
$Y_1, Y_2 \geq 0$

Solution of primal	*solution of dual*
Writing the problem in standard simplex form : $Max\ Z = 40X_1 + 50X_2 + \mathbf{0S_1} + \mathbf{0S_2}$ Subject to $2X_1 + 3X_2 + S_1 = 3$ $8X_1 + 4X_2 + S_2 = 5$ $X_1, X_2, S_1, S_2 \geq 0$	*Writing the problem in standard simplex form :* $Max\ W' = -3Y_1 - 5Y_2 + 0S_1 + 0S_2$ *where* $W' = -W$ *subject to* $2Y_1 + 8Y_2 - S_1 + A_1 = 40$ $3Y_1 + 4Y_2 - S_2 + A_2 = 50$ $Y_1, Y_2, S_1, S_2, A_1, A_2 \geq 0$

First we start with primal form , let $X_1 = X_2 = 0$ this implies that the initial basic feasible solution is $S_1 = 3$ *and* $S_2 = 5$. Construct the simplex table

Basic variables	C_B	X_B	X_1	X_2	Basis Matrix		Min Ratio X_B/X_k
		C_j	40	50	0	0	
					S_1	S_2	
S_1	0	3	2	$[\mathbf{3}]$	1	0	$\mathbf{\frac{3}{3} = 1\ Min} \leftarrow$
$\mathbf{S_2}$	0	5	**8**	4	0	1	$\frac{5}{4} = 1\frac{1}{4}$
	$Z = C_B X_B = 0$		$\Delta_1 = -40$	$\Delta_2 = \mathbf{-50}$ ↑	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = Z_j - C_j = C_B X_j - C_j$

Basic variables	C_B	X_B	X_1	X_2	Basis Matrix		Min Ratio X_B/X_k
		C_j	40	50	0	0	
					S_1	S_2	
X_2	50	1	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{\frac{2}{3}} = \frac{3}{2}$
$\mathbf{S_2}$	0	1	$\left[\mathbf{\frac{16}{3}}\right]$	0	$-\frac{4}{3}$	1	$\mathbf{\frac{1}{\frac{16}{3}} = \frac{3}{16}\ Min} \leftarrow$
$X_1 = S_1 = 0$	Z=$C_B X_B$=50		$\mathbf{-\frac{20}{3}}$ ↑	*0*	$\frac{50}{3}$	0	$\Delta_j = C_B X_j - C_j$
X_2	50	$\frac{7}{8}$	0	**1**	$\frac{1}{2}$	$-\frac{1}{8}$	
$\mathbf{X_1}$	40	$\frac{3}{16}$	1	0	$-\frac{1}{4}$	$\frac{3}{16}$	
$S_1 = S_2 = 0$	Z=$C_B X_B$=51.25		0	*0*	**15**	$\mathbf{\frac{5}{4}}$	$\Delta_j = C_B X_j - C_j$

Since all Δ_j are positive, hence the optimal solution of the primal is given by

$$X_1 = \frac{3}{16}\ , \qquad X_2 = \frac{7}{8}\ , and\ Z = \left(\frac{3}{16} \times 40\right) + \left(\frac{7}{8} \times 50\right) = 51.25$$

Now, we solve the dual problem to get the optimal solution which is same as the solution of primal problem.

Phase 1. Assigning a cost -1 to artificial variables A_1 and A_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes*:*

$$Max\ W'^{*} = 0Y_1 + 0Y_2 + 0S_1 + 0S_2 - 1A_1 - 1A_2$$

Subject to the constraints

$$2Y_1 + 8Y_2 - S_1 + A_1 = 40$$
$$3Y_1 + 4Y_2 - S_2 + A_2 = 50$$
$$Y_1, Y_2, S_1, S_2, A_1, A_2 \geq 0$$

Let $Y_1 = Y_2 = S_1 = S_2 = 0$ this implies that the initial basic feasible solution is $A_1 = 40$ and $A_2 = 50$, Construct the simplex table.

		C_j	0	0	0	0	-1	-1	
Basic variables	C_B	Y_B	Y_1	Y_2	S_1	S_2	A_1	A_2	Min. Ratio Y_B/Y_k
A_1	-1	40	2	**[8]**	-1	0	1	0	$40/8 = 5$ ←
A_2	-1	50	3	4	0	-1	0	1	$50/4 = 12.5$
	$W'^{*} = -90$		-5	-12^{*} ↑	1	1	0	0	← Δ_j
Y_2	0	5	$1/4$	1	$-1/8$	0	$1/8$	0	$\frac{5}{\frac{1}{4}} = 20$
A_2	-1	30	**[2]**	0	$1/2$	-1	$-1/2$	1	$30/2 = 15$ ←
	$W'^{*} = -30$		-2^{*} ↑	0	$-1/2$	1	$3/2$	0	
Y_2	0	$5/4$	0	1	$-3/16$	$1/8$	$3/16$	$-1/8$	
Y_1	0	15	1	0	$1/4$	$-1/2$	$-1/4$	$1/2$	
	$W'^{*} = 0$		0	0	0	0	1	1	← $\Delta_j \geq 0$

Since All $\Delta_j \geq 0$ and no artificial variable appears in the basic, an optimal solution to the auxiliary problem has been attained.

Phase 2. In this phase now consider the actual costs associated with original variables, the objective function thus becomes $Max\ W' = -3Y_1 - 5Y_2 + 0S_1 + 0S_2$.

Now apply simplex method in the usual manner.

		C_j	-3	-5			
Basic variables	C_B	Y_B	Y_1	Y_2	S_1	S_2	Min. Ratio $^{Y_B}/_{Y_k}$
Y_2	-5	$^5/_4$	0	1	$-^3/_{16}$	$^1/_8$	
Y_1	-3	15	1	0	$^1/_4$	$-^1/_2$	
	$W' = -51.25$		0	0	$\mathbf{^3/_{16}}$	$\mathbf{^7/_8}$	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimal basic feasible solution has been attained.
Hence optimal solution of the dual is given by : $Y_1 = 15,\ \ Y_2 = {^5/_4},\ \ Min\ Z = 51.25$

Conclusion. From above comparison, it is concluded that the solution to a primal problem of linear programming can always provide a solution to its dual.

Homework: Use duality to solve :

1.
$Min\ Z = 3X_1 + X_2$
Subject to constraints
$X_1 + X_2 \geq 1$
$2X_1 + 3X_2 \geq 2$
$X_1, X_2 \geq 0$

2. $Min\ Z = 2500X_1 + 3000X_2$
Subject to constraints
$X_1 \geq 30$
$X_2 \geq 20$
$X_1 + X_2 \geq 60$
$X_1, X_2 \geq 0$

3. $Min\ Z = X_1 - X_2$
Subject to
$2X_1 + X_2 \geq 2$
$-X_1 - X_2 \geq 1$
$X_1, X_2 \geq 0$

6. The Transportation problem

Definition. The transportation Problem is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

For example, a tyre manufacturing concern has m factories located in **m** different cities. The total supply potential of manufactured product is absorbed by **n** retail dealers in **n** different cities of the country. Then, transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factories locations to various retail dealers.

6.1. Mathematical Formulation

Let there be m origins, $\boldsymbol{ith}$ origin possessing $\boldsymbol{a_i}$ units of a certain product, whereas there are **n** destinations (**n** may or may not be equal to **m**)with destination j requiring $\boldsymbol{b_j}$ units. Costs of shipping of an item from each of m origins (sources) to each of the n destinations are known either directly or indirectly in terms of mileage, shipping hours, etc. let $\boldsymbol{c_{ij}}$ be the cost of shipping one unit product from $\boldsymbol{ith}$ origin (source) to $\boldsymbol{jth}$ destination, and '$\boldsymbol{x_{ij}}$' be the amount to be shipped from the $\boldsymbol{ith}$ origin to $\boldsymbol{jth}$ destination.

It is also assumed that total availability $\sum a_i$ satisfies the total requirements $\sum b_j$, that is,

$\sum a_i = \sum b_j \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$
(1)

(In case $\sum a_i \neq \sum b_j$ some manipulation is required to make $\sum a_i = \sum b_j$, which will be shown later). The problem now is to determine non-negative (≥ 0) values of 'x_{ij}' satisfying both, availability constraints:

$\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1, 2, \dots, m$
(2)

As well as the requirement constraints

$\sum_{i=1}^{m} x_{ij} = b_j$ for $j = 1, 2, \dots, n$
(3)

And minimizing the total cost of transportation (shipping)

$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij}$ (objective function).
(4)

Remark:

1.constraints equations (2), (3) and the objective function (4) are all linear in x_{ij}, so it may be looked like a linear programming problem. This special type of linear programming problem will be called a transportation problem(T.P.)

2. By requiring strict inequalities $a_i > 0 \ and \ b_j > 0$ we are not restricting anything. Since all $x_{ij} \geq 0$, it follows that each $a_i \geq 0 \ and \ each \ b_i \geq 0$. Moreover, any $a_k = 0 \Rightarrow x_{kj} = 0$ and thus can be eliminated from the problem.

6.2. Feasible Solution, Basic Feasible Solution, And Optimum Solution

The terms feasible solution, basic feasible solution and optimum solution may be formally defined with reference to the transportation problem (T.P.) as follows:

1.**Feasible solution (FS)**. A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called ***a feasible solution.***

2. **Basic feasible solution (BFS)**. A feasible solution to a m-origin, n-destination problem is said to be basic if the number of positive allocations are $\boldsymbol{m + n - 1}$, that is, one less than sum of rows and columns.

If the number of allocations in a basic feasible solution are less than $\boldsymbol{m + n - 1}$, it is called degenerate BFS (otherwise, non-degenerate BFS).

3. **Optimal Solution.** A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

6.2.1. Existence of Feasible Solution

Theorem 7.1. (Existence of Feasible Solution). A necessary and sufficient condition for the existence of feasible solution of a transportation problem is $\sum a_i = \sum b_j \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$.

6.2.2. Basic Feasible Solution of Transportation Problem

It has been observed that a transportation problem is a special case of a linear programming problem. So a basic feasible solution of a transportation problem has the same definition as earlier given for linear programming problem. However we observed that in the case of a transportation problem, there are only $\boldsymbol{m + n - 1}$ basic variables out of mn unknown. This happens due to redundancy in the constraints of the transportation problem. This can be easily justified by the following theorem.

Theorem 7.2.

The number of basic variable in a transportation problem are at the most $\boldsymbol{m + n - 1.}$

Remark:

It is concluded that a basic feasible solution will consist of at most $\boldsymbol{m + n - 1}$ positive variables, others being zero. In the degenerate case, some of the basic variables will also be zero, that is, the number of positive variables will now become less than $\boldsymbol{m+n-1}$. By fundamental theorem of linear programing, one of the basic feasible solution will be the optimal solution.

6.2.3 Existence of Optimal Solution

Theorem 7.3. (Existence of Optimal Solution). There always exists an optimal solution to a balanced transportation problem.

6.3. Tabular representation

Suppose there are m factories and n warehouse. The transportation problem is usually represented in a tabular form. Calculating are made directly on the transportation arrays which give the current trial solution.

Warehouse→ / Factories ↓	W_1	W_2	$\cdots$	W_j	$\cdots$	W_n	Factory Capacity
F_1	C_{11} X_{11}	C_{12} X_{12}	$\cdots$	C_{1j} X_{1j}	$\cdots$	C_{1n} X_{1n}	a_1
F_2	C_{21} X_{21}	C_{22} X_{22}	$\cdots$	C_{2j} X_{2j}	$\cdots$	C_{2n} X_{2n}	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F_i	C_{i1} X_{i1}	C_{i2} X_{i2}	$\cdots$	C_{ij} X_{ij}	$\cdots$	C_{in} X_{in}	a_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
F_m	C_{m1} X_{m1}	C_{m2} X_{m2}	$\cdots$	C_{mj} X_{mj}	$\cdots$	C_{mn} X_{mn}	a_m
Warehouse requirements	b_1	b_2	$\cdots$	b_j	$\cdots$	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Remark: The product $X_{ij}(C_{ij})$ gives the net cost of shipping X_{ij} unit from factory F_i to warehouse W_j.

6.4. Method for Initial Basic Feasible Solution to a transportation problems

Some simple methods are described here to obtain the initial basic feasible solution of the transportation problem.

1. North-West Corner Rule
2. Least Cost Method
3. Vogel's Approximation Method

1. North-West Corner Rule

Step1. The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is, $X_{11} = \min(a_1, b_1)$. This value of X_{11} is then entered in the cell (1,1) of the transportation table.

Step2. (a) If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation amount $X_{21} = \min(a_2, b_1 - X_{11})$ in the cell (2,1)

(b) If $b_1 < a_1$, move horizontally right-side to the second column and make the second allocation of amount $X_{12} = \min(a_1 - X_{11}, b_2)$ in the cell (1,2).

(c) If $b_1 = b_2$, there is a tie for the second allocation. One can make the second allocation of magnitude $X_{12} = \min(a_1 - a_1, b_2) = 0$ in the cell (1,2) or $X_{21} = \min(a_2, b_1 - b_1)$ in the cell (2,1).

Step3. Start from the new north-west corner of the transportation table and repeat **steps 1** and **2** until all the requirements are satisfied.

Example 1. Use North-West Corner Rule to find the initial basic feasible solution of the following transportation problem.

Warehouse→ / Factories ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	10	0	20	11	15
F_2	12	7	9	20	25
F_3	0	14	16	18	5
Warehouse requirements	5	15	15	10	

Solution. Sum of availabilities $\sum a_{ij} = 15 + 25 + 5 = 45$

Sum of requirements $\sum b_j = 5 + 15 + 15 + 10 = 45$

Since $\sum a_i = \sum b_j$, the necessary and sufficient condition is satisfied and hence there exists a solution to the given transportation problem.
We see that the number of occupying cells are $m + n - 1 = 3 + 4 - 1 = 6$,
That is , **the basic variables** are **6** only and **non-basic variables** are **6** out of **12** variables.
$X_{11} = \min(a_1, b_1) = \min(5,15) = 5$, remove first column and move horizontally
$X_{12} = \min(a_1 - X_{11}\,, b_2) = (10\,, 15) = 10$, remove first row
$X_{22} = \min(a_2\,, b_2 - X_{12}) = (25,15 - 10) = (25,5) = 5$, remove second column
$X_{23} = \min(a_2 - X_{22}\,, b_3) = (25 - 5\,, 15) = 15$, remove third column
$X_{24} = \min(a_2 - (x_{22} + X_{23}\,, b_4) = (25 - 20\,, 10) = 5$, removed third row
$X_{34} = 5$

Warehouse→ / Factories ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	[10] $X_{11} = 5$	[0] $X_{12} = 10$	[20] $X_{13} = 0$	[11] $X_{14} = 0$	15
F_2	[12] $X_{21} = 0$	[7] $X_{22} = 5$	[9] $X_{23} = 15$	[20] $X_{24} = 5$	25
F_3	[0] $X_{31} = 0$	[14] $X_{32} = 0$	[16] $X_{33} = 0$	[18] $X_{34} = 5$	5
Warehouse requirements	5	15	15	10	45 / 45

Basic variables are $X_{11} = 5$, $X_{12} = 10$, $X_{22} = 5$, $X_{23} = 15$, $X_{24} = 5$, $X_{34} = 5$
Non-basic variables are $X_{13} = X_{14} = X_{21} = X_{31} = X_{32} = X_{33} = 0$
The total cost to transportation is $Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij}X_{ij}$

$$Z = 5 \times 10 + 10 \times 0 + 5 \times\times 7 + 15 \times 9 + 5 \times 20 + 5 \times 18 = 410$$

The number of occupying cells are $m + n - 1 = 3 + 4 - 1 = 6$

Homework. Use North-West Corner Rule to find the initial basic feasible solution of the following transportation problem.

1.

Warehouse→ / Factories ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	[10]	[0]	[20]	[11]	10
F_2	[12]	[7]	[9]	[20]	5

F_3	0	14	16	18	15
Warehouse requirements	5	10	8	7	

2.

Warehouse→ / Factories ↓	W_1	W_2	W_3	W_4	W_5	Factory Capacity
F_1	37	27	28	34	30	100
F_2	29	32	32	27	28	125
F_3	34	27	37	30	30	150
Warehouse requirements	75	60	70	10	90	

3.

	1	2	3	Productive Quantities
1	1	0	2	4
2	3	5	4	6
3	1	2	3	10
Demand Quantities	3	5	12	

2. Least Cost Method

Step1. Determine the smallest cost in the cost matrix of the transportation table. Let it be (C_{ij}). Allocate $X_{ij} = \min(a_i, b_j)$ in the cell (i , j).

Step2. (1) If $X_{ij} = a_i$, cross-out the ith row of the transportation table and decrease b_j by a_i. Go to step 3.

(2) If $X_{ij} = b_j$, cross-out the jth column of the transportation table and decrease a_i by b_j. Go to step 3.

(3) If $X_{ij} = a_i = b_j$, cross-out either the ith row or jth $column$ but not both.

Step3. Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost in not unique, make an arbitrary choice among the minima.

Example2. Use north-west corner rule and least cost method to find the initial basic feasible solution of the following transportation problem and compare between the two methods.

	D_1	D_2	Supply
S_1	4	2	60
S_2	7	5	40
S_3	3	10	70
Demand	105	65	

Solution.

First, solve by north-west corner rule.

sum of availabilities $\sum a_{ij} = 60 + 40 + 70 = 170$

Sum of requirements $\sum b_j = 105 + 65 = 170$

Since $\sum a_i = \sum b_j$, the necessary and sufficient condition is satisfied and hence there exists a solution to the given transportation problem.

We see that the number of occupying cells are $m + n - 1 = 3 + 2 - 1 = 4,$

That is , **the basic variables** are four only and **non-basic variables** are two out of six variables.

$X_{11} = \min(a_1, b_1) = \min(60{,}105) = 60$, remove first row and move vertically

$X_{21} = \min(a_2 , b_1 - X_{11}) = (40 , 105 - 60) = 40$, remove second row and remaining only 15 in first column.

$X_{31} = \min(a_3 , b_1 - (X_{11} - X_{21}) = (70{,}105 - 100) = (70{,}5) = 5$, remove first column

$X_{32} = 65$

	D_1	D_2	supply
S_1	4 $X_{11} = 60$	2 $X_{12} = 0$	60
S_2	7 $X_{21} = 40$	5 $X_{22} = 0$	40
S_3	3 $X_{31} = 5$	10 $X_{32} = 65$	70
Demand	105	65	

Basic variables are $X_{11} = 60$, $X_{21} = 40$, $X_{31} = 5$, $X_{32} = 65$

Non-basic variables are $= X_{12} = X_{22} = 0$

The total cost to transportation is $Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$

$$Z = 60 \times 4 + 40 \times 7 + 5 \times 3 + 65 \times 10 = 1185$$

The number of occupying cells are $m + n - 1 = 3 + 2 - 1 = 4$

-Second, solve by least cost method

We choose the smallest cost in the matrix. It is in the cell (1 ,2).

$\boldsymbol{X_{12}} = \min(a_1 , b_2) = (60 ,65) = 60$, **remove first row** and remain only **5** in second column we choose the second smallest cost in the matrix. It is in the cell (3,1).

$\boldsymbol{X_{31}} = \min(a_3 , b_1) = (70 , 105) = 70$, **remove third row** and remain 35 in first column.

We choose the third smallest cost in the matrix. It is in the cell (2,2).

$\boldsymbol{X_{22}} = \min(a_2 , b_2) = (40 ,5) = 5$, **remove second column**

The last cost remains in the cell (2 ,1) $which\ is\ X_{21} = 35$

	D_1	D_2	Supply
S_1	[4] $X_{11} = 0$	[2] $\boldsymbol{X_{12} = 60}$	60
S_2	[7] $X_{21} = 35$	[5] $X_{22} = 5$	40
S_3	[3] $\boldsymbol{X_{31} = 70}$	[10] $X_{32} = 0$	70
Demand	105	65	170 / 170

Basic variables are $X_{12} = 60$, $X_{21} = 35$, $X_{22} = 5$, $X_{31} = 70$

Non-basic variables are $= X_{11} = X_{32} = 0$

The total cost to transportation is $Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$

$$Z = 60 \times 2 + 5 \times 5 + 35 \times 7 + 70 \times 3 = 600$$

The number of occupying cells are $m + n - 1 = 3 + 2 - 1 = 4$

We see that the cost in the least cost method is less than in the north-west corner method.

Homework. Use least cost method and north-west corner rule to find the initial basic feasible solution of the following transportation problem and compare between two solutions .

Branches / Factories	A	B	C	Factory capacity
1	[1]	[0]	[2]	4
2	[3]	[5]	[4]	6
3	[1]	[2]	[3]	10
Demand each branch	3	5	12	

3. Vogel's Approximation Method

Step1. For each row of the transportation table identity the smallest and next-to-smallest Cost. Determine the difference between them for each row. These are called 'penalties'. Put them along side the transportation table by enclosing them in the parentheses against the respective rows. Similarly, compute these penalties for each column.

Step2. Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the largest penalty correspond to ith row and let C_{ij} be the smallest cost in the ith row. Allocate the largest possible amount $X_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross-out the ith row or the jth column in the usual manner.

Step3. Again compute the column and row penalties for the reduced transportation table and then go to step2. Repeat the procedure until all the requirements are satisfied.

Remark. By saying "cross-out a row or a column" we shall mean that no cells from that row or column can be chosen for the basis entry at a later step.

Example3. Use Vogel's Approximation Method to find the initial basic feasible solution of the following transportation problem.

	D_1	D_2	supply
S_1	4	2	60
S_2	7	5	40
S_3	3	10	70
Demand	105	65	170 / 170

Solution. sum of availabilities $\sum a_{ij} = 60 + 40 + 70 = 170$

Sum of requirements $\sum b_j = 105 + 65 = 170$

Since $\sum a_i = \sum b_j$, the necessary and sufficient condition is satisfied and hence there exists a solution to the given transportation problem.

(1) Calculate the difference between the two smallest costs in each row and column in the given table.

	D_1	D_2	supply	**Difference between the smallest two costs (penalty)**
S_1	4	2	60	(2)
S_2	7	5	40	(2)
S_3	3	10	70	**(7)**
Demand	105	65	170 / 170	
Difference between the smallest two costs (penalty)	**(1)**	**(3)**		

2. We choose the largest penalty among all the rows and columns which correspond to third row.

3. **We choose the smallest cost in the** $\boldsymbol{third\ row}$.

The smallest cost is 3 which corresponds to X_{31}, that is,

$X_{31} = \min(a_3\,, b_1) = (70\,, 105) = 70$, remove the third row and remaining only 35 in first column

	D_1	D_2	supply	**Difference between the smallest two costs (penalty)**
S_1	4	2	60	(2)
S_2	7	5	40	(2)
S_3	3 $\boldsymbol{X_{31} = 70}$	10	70	**(7)**←
Demand	105	65	170 / 170	
Difference between the smallest two costs (penalty)	**(1)**	**(3)**		

Again compute the column and row penalties for the reduced transportation table and then go to step2. Repeat the procedure until all the requirements are satisfied.

	D_1	D_2	supply	Difference between the smallest two costs (penalty)
S_1	4	2 $X_{12} = 60$	60	(2)←
S_2	7	5	40	(2)
Demand	105	65		
Difference between the smallest two costs (penalty)	**(3)**	**(3)**		

$X_{12} = \min(a_1, b_1) = (60\,, 65\,) = 60$, remove the first row and remaining only 5 in the first column.

	D_1	D_2	supply
S_2	7 $X_{21} = 35$	5 $X_{22} = 5$	40

$$X_{21} = 35 \; and \; X_{22} = 5$$

We return to the basic transportation problem to distribute all the quantities in the cells for the origin problem.

	D_1	D_2	supply
S_1	4 $X_{11} = 0$	2 $X_{12} = 60$	60
S_2	7 $X_{21} = 35$	5 $X_{22} = 5$	40
S_3	3 X_{31}70	10 $X_{32} = 0$	70
Demand	105	65	170 / 170

Basic variables are $X_{12} = 60$, $X_{21} = 35$, $X_{22} = 5$, $X_{31} = 70$
Non-basic variables are $= X_{11} = X_{32} = 0$
The total cost to transportation is $Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$
$Z = 60 \times 2 + 5 \times 5 + 35 \times 7 + 70 \times 3 = 600$
The number of occupying cells are $m + n - 1 = 3 + 2 - 1 = 4$

Example3. Use Vogel's Approximation Method to find the initial basic feasible solution of the following transportation problem.

Warehouse→ / Factories ↓	D_1	D_2	D_3	D_4	Factory Capacity
S_1	20	22	17	4	120
S_2	24	37	9	7	70
S_3	32	37	20	15	50
Warehouse requirements	60	40	30	110	240 / 240

Solution.

	D_1	D_2	D_3	D_4	**supply**	**Difference between the smallest two costs (penalty)**
S_1	20	22 $X_{12} = 40$	17	4	120	(13)
S_2	24	37	9	7	70	(2)
S_3	32	37	20	15	50	(5)
Demand	60	40	30	110	240 / 240	
Difference between the smallest two cost (penalty)	(4)	↑**(15)**	(8)	(3)		

$X_{12} = \min(120,40) = 40$, **remove the second column** from the table and remaining only 80 in the first row

	D_1	D_3	D_4	supply	Difference between the smallest two costs (penalty)
S_1	20	17	4 $X_{14} = 80$	~~120~~ 80	**(13)←**
S_2	24	9	7	70	(2)
S_3	32	20	15	50	(5)
Demand	60	30	110	240 / 240	
Difference between the smallest two costs (penalty)	(4)	(8)	(3)		

$X_{14} = \min(80{,}110) = 80$, **remove the first row** and remaining only 30 in the fourth column

	D_1	D_3	D_4	supply	Difference between the smallest two costs (penalty)
S_2	24	9 $\boldsymbol{X_{23} = 30}$	7	70	(2)
S_3	32	20	15	50	(5)
Demand	60	30	~~110~~ 30	240 / 240	
Difference between the smallest two costs (penalty)	(8)	(**11**) ↑	(8)		

$X_{23} = \min(70\ ,30) = 30$**, remove third column three** and remaining only 40 in second row

	D_1	D_4	supply	Difference between the smallest two costs (penalty)
S_2	24 $X_{21} = 10$	7 $X_{24} = 30$	~~70~~ 40	**(17)←**
S_3	32 $X_{31} = 50$	15	50	(17)
Demand	60	~~110~~ 30	240 / 240	
Difference between the smallest two costs (penalty)	(8)	(8)		

$X_{24} = \min(70, 30) = 30$, **remove fourth column** and remaining only **40** in first row.

$$X_{21} = 10 \ , \qquad X_{31} = 50$$

We will return to the origin table of transportation problem.

Warehouse→ / Factories ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	20 $X_{11} = 0$	22 $X_{12} = 40$	17 $X_{13} = 0$	4 $X_{14} = 80$	120
F_2	24 $X_{21} = 10$	37 $X_{22} = 0$	9 $X_{23} = 30$	7 $X_{24} = 30$	70
F_3	32 $X_{31} = 50$	37 $X_{32} = 0$	20 $X_{33} = 0$	15 $X_{34} = 0$	50
Warehouse requirements	60	40	30	110	240 / 240

The total cost to transportation is $Min\ Z = \sum_{j=1}^{n} \ \sum_{i=1}^{m} C_{ij} X_{ij}$

$$Z = 40 \times 22 + 80 \times 4 + 10 \times 24 + 30 \times 9 + 30 \times 7 + 50 \times 32 = 3520$$

The number of occupying cells are $m + n - 1 = 3 + 4 - 1 = 6$

Homework.

(1) The following table explains transportation problem that contains three units and three centers and also explains available quantities in the units and demand quantities from centers. It also explains transport cost from units to centers.

Find the initial basic feasible solution by the following three method and compare the results by these three methods

1. By north-west corner rule.
2. By least cost method.
3. By Vogel's approximation method.

Centers / **Units**	***A***	***B***	**C**	**Total available quantities**
1	10	2	16	24
2	0	8	4	28
3	14	12	6	8
Total demand quantities	18	20	22	60 / 60

7.Testing initial basic feasible solution and obtain by it the optimal solution

To find the optimal solution there are two methods.

1.Stepping Stone Method:

2. Modified Distribution method

1.Stepping Stone Method: is an optimization technique used to find optimal transformation cost.

In stepping stone method, we form loops for every unoccupied cell and evaluate them for optimality.

Steps in Stepping Stone Method:

1. Determine an initial basic feasible solution using any one of the following:

a)North-West Corner Rule

b)Least cost Mothed

c) **Vogel's Approximation Method**

2. Make sure that the number of occupied cells is exactly equal to m+n-1, where m is the number of rows and n is the number of columns.

3. Select an unoccupied cell. Beginning at this cell, trace a closed path, starting from the selected unoccupied cell until finally returning to that same unoccupied cell.

The cells at the turning points are called "Stepping Stones" on the path.

4. Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, beginning with the plus sign at unoccupied cell to be evaluated.

5. Add the unit transportation costs associated with each of the cell traced in the closed path. This will give net change in terms of cost.

6. Repeat steps 3 to 5 until all unoccupied cells are evaluated.

7. Check the sign of each of the net change in the unit transportation costs. If all the net changes computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease the total transportation cost, so move to step 8..

8. Select the unoccupied cell having the most negative net cost change and determine the maximum number of units that can be assigned to this cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the

entering cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.

Example 4. Find the optimal solution by stepping stone method for the following T.P.

	D_1	D_2	supply
S_1	4 $X_{11} = 60$	2 $X_{12} = 0$	60
S_2	7 $X_{21} = 40$	5 $X_{22} = 0$	40
S_3	3 $X_{31} = 5$	10 $X_{32} = 65$	70
Demand	105	65	

solution. (1)The initial basic feasible solution which obtained by north-west corner rule is

$$Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

$$Z = 60 \times 4 + 40 \times 7 + 5 \times 3 + 65 \times 10 = 1185$$

(2) the number of occupied cells is exactly equal to m+n-1, That is $3 + 2 - 1 = 4$, this emphasis that the solution can be improved and reach by it to the optimal solution.

Which are $X_{11} = 60,\ X_{21} = 40,\ X_{31} = 5,\ X_{32} = 65.$

(3) Select an unoccupied cell which is $X_{12} = 0$, trace a closed path, starting from the selected unoccupied cell until finally returning to that same unoccupied cell.

(1) $S_1D_2 \rightarrow S_1D_1 \rightarrow S_3D_1 \rightarrow S_3D_2$

4. Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, beginning with the plus sign at unoccupied cell to be evaluated.

$$2 - 4 + 3 - 10 = -9$$

5. Add the unit transportation costs associated with each of the cell traced in the closed path. This will give net change in terms of cost. Let the quantity which it will convert through unoccupied cell is K unit.

	D_1	D_2	supply
S_1	$60 - K$	K	60
S_2			40
S_3	$5 + K$	$65 - K$	70
Demand	105	65	

$K = \text{Min}(60,65) = 60$

6. Add the unit transportation costs associated with each of the cell traced in the closed path. This will give net change in terms of cost.

	D_1	D_2	supply
S_1	4 $X_{11} = 0$	2 $X_{12} = 60$	60
S_2	7 $X_{21} = 40$	5 $X_{22} = 0$	40
S_3	3 $X_{31} = 65$	10 $X_{32} = 5$	70
Demand	105	65	

$$Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij} = 60 \times 2 + 40 \times 7 + 65 \times 3 + 5 \times 10 = 645$$

7. Repeat steps 3 to 5 until all unoccupied cells are evaluated.

(2) $S_2D_2 \rightarrow S_2D_1 \rightarrow S_3D_1 \rightarrow S_3D_2$,

$5 - 7 + 3 - 10 = -9$

	D_1	D_2	supply
S_1			60
S_2	$40 - K$	K	40
S_3	$65 + K$	$5 - K$	70
Demand	105	65	

$K = \text{Min}(40,5) = 5$

	D_1	D_2	supply
S_1	4 $X_{11} = 0$	2 $X_{12} = 60$	60
S_2	7 $X_{21} = 35$	5 $X_{22} = 5$	40
S_3	3 $X_{31} = 70$	10 $X_{32} = 0$	70
Demand	105	65	

$$Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij} = 60 \times 2 + 35 \times 7 + 5 \times 5 + 70 \times 3 = 600$$

Example 5. Find the optimal solution by stepping stone method for the following T.P.

Destination→ / Sources ↓	D_1	D_2	D_3	D_4	D_5	Capacity
S_1	37 $X_{11} = 0$	27 $X_{12} = 30$	28 $X_{13} = 70$	34 $X_{14} = 0$	30 $X_{15} = 0$	100
S_2	29 $X_{21} = 45$	32 $X_{22} = 0$	32 $X_{23} = 0$	27 $X_{24} = 80$	28 $X_{25} = 0$	125
S_3	34 $X_{31} = 30$	27 $X_{32} = 30$	37 $X_{33} = 0$	30 $X_{34} = 0$	30 $X_{35} = 90$	150
Demand	75	60	70	80	90	

solution. (1)The initial basic feasible solution which obtained by north-west corner rule is

$$Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

$$Min\ Z = 30 \times 27 + 70 \times 28 + 45 \times 29 + 80 \times 27 + 30 \times 34 + 30 \times 27 + 90 \times 30 = 10765$$

The number of occupied cells is exactly equal to m+n-1, That is $3 + 5 - 1$ =7, this emphasis that the solution can be improved and reach by it to the optimal solution.

1. We start by testing empty cell $S_1 D_1$

$$S_1 D_1 \to S_1 D_2 \to S_3 D_2 \to S_3 D_1$$

$$37 - 27 + 27 - 34 = 3$$

Since the result change of the cost is positive then the solution for this cell is optimal, that is, it must be empty.

2. Test the empty cell S_1D_4

$$S_1D_4 \rightarrow S_2D_4 \rightarrow S_2D_1 \rightarrow S_3D_1 \rightarrow S_3D_2 \rightarrow S_1D_2$$

$34 - 27 + 29 - 34 + 27 - 27 = 2$, since the result is positive , then the solution is optimal for this cell.

3. By same way , we test the cell S_1D_5

$$S_1D_5 \rightarrow S_3D_5 \rightarrow S_3D_2 \rightarrow S_1D_2$$

$30 - 30 + 27 - 27 = 0$, since the result is zero , then the solution is optimal for this cell.

4. Test the cell S_2D_2

$$S_2D_2 \rightarrow S_2D_1 \rightarrow S_3D_1 \rightarrow S_3D_2$$

$32 - 29 + 34 - 27 = 10$, since the result is positive , then the solution is optimal for this cell.

5. Test the cell S_2D_3 and the path is

$$S_2D_3 \rightarrow S_2D_1 \rightarrow S_3D_1 \rightarrow S_3D_2 \rightarrow S_1D_2 \rightarrow S_1D_3$$

$32 - 29 + 34 - 27 + 27 = 9$, since the result is positive , then the solution is optimal for this cell.

6. Test the cell S_2D_5 and the path is

$$S_2D_5 \rightarrow S_3D_5 \rightarrow S_3D_1 \rightarrow S_2D_1$$

28-30+34-29=3 , since the result is positive , then the solution is optimal for this cell.

7. Test the cell S_3D_3 and the path is

$$S_3D_3 \rightarrow S_3D_2 \rightarrow S_1D_2 \rightarrow S_1D_3$$

$37 - 27 + 27 - 28 = 9$, since the result is positive , then the solution is optimal for this cell.

8. Test the cell S_3D_4 and the path is

$$S_3D_4 \rightarrow S_3D_1 \rightarrow S_2D_1 \rightarrow S_2D_4$$

$30 - 34 + 29 - 27 = -2$, here the result is negative , this mean that the solution is not optimal for this cell and hence it must be occupied.

Destination→ / Sources ↓	D_1	D_2	D_3	D_4	D_5	Capacity
S_1						100
S_2	$45+K$ →			→ $80-K$ ↓		125
S_3	$30-k$ ↑			K ←		150
Demand	75	60	70	10	90	

$$K = \text{Min}(30{,}80) = 30$$

Destination→ / Sources ↓	D_1	D_2	D_3	D_4	D_5	Capacity
S_1	[37] $X_{11}=0$	[27] $X_{12}=30$	[28] $X_{13}=70$	[34] $X_{14}=0$	[30] $X_{15}=0$	100
S_2	[29] $X_{21}=75$	[32] $X_{22}=0$	[32] $X_{23}=0$	[27] $X_{24}=50$	[28] $X_{25}=0$	125
S_3	[34] $X_{31}=0$	[27] $X_{32}=30$	[37] $X_{33}=0$	[30] $X_{34}=30$	[30] $X_{35}=90$	150
Demand	75	60	70	80	90	

$$Min\ Z = \sum_{i=1}^{3} \sum_{j=1}^{5} C_{ij} X_{ij} = 10705$$

8. Unblanced Transportation Problems

So far we have discussed the balanced type of transportation problems where the total destination requirements equals the total original capacity (this mean that, $\sum a_{ij} = \sum bij$) . But, sometimes in particular situations, the demand may be more than the availability or vice versa (this mean that, $\sum a_{ij} \neq \sum bij$).

Thus, if an transportation problem, the sum of all available quantities is not equal to the sum of requirements, that is, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then such problem is called unbalanced transportation problem.

7.6.1. To modify unbalanced T.P. to balanced type

An unbalanced T.P. may occur in two different cases .

Case 1. Excess of availability, this mean that $\sum a_i \geq \sum b_j$,

Case 2. Shortage in availability, this mean that $\sum a_i \leq \sum b_j$

Working Rule 1: Whenever $\sum \boldsymbol{a_i} \geq \sum \boldsymbol{b_j}$, we introduce a dummy destination-column in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to the difference $\sum a_i - \sum b_j$.

Working Rule 2: Whenever $\sum \boldsymbol{a_i} \leq \sum \boldsymbol{b_j}$, introduce a dummy source in the transportation table. The cost of transportation form this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference $(\sum b_j - \sum a_i)$.

Thus, an unbalanced transportation problem can be modified to balanced problem by simply introducing fictitious sink in the first case a and fictitious source in the second. The inflow from the source to a fictitious sink represents the surplus at the source. Similarly, the flow from the fictitious source to sink represents the unfilled demand at the sink. For convenience, costs of transporting a unit item from fictitious sources or to fictitious sinks 9as the case may be) are assumed to be zero. The resulting problem then becomes balanced one and can be solved by the same procedure as explained earlier.

Example 6. find the initial basic feasible solution of the following transportation problem by any method.

Centers / **Units**	$\boldsymbol{D_1}$	$\boldsymbol{D_2}$	$\boldsymbol{D_3}$	**Total available quantities**
S_1	2	1	2	20
S_2	1	2	3	9
S_3	4	2	1	11
Total demand quantities	10	8	15	40 / 33

Solution: Since $\sum \boldsymbol{a_i} = \mathbf{40} > \sum \boldsymbol{b_j} = \mathbf{33}$, that is, the given T.P. problem is **unbalanced.** We introduce a dummy destination-column $\boldsymbol{D_4}$ in the transportation table. The unit transportation costs to this dummy destination are all set equal to zero. The requirement at this dummy destination is assumed to be equal to $40 - 33 = 7$.

Centers \ **Units**	$\boldsymbol{D_1}$	$\boldsymbol{D_2}$	$\boldsymbol{D_3}$	$\boldsymbol{D_4}$	**Total available quantities**
S_1	2	1	2	0	20
S_2	1	2	3	0	9
S_3	4	2	1	0	11
Total demand quantities	10	8	15	7	40 / 40

We will solve it by least cost method.

We choose the smallest cost in the matrix. It is in the cell (1 ,2).
$\boldsymbol{X_{12}} = \min(a_1 , b_2) = (20 ,8) = 8$, **remove second column** and remain only **12** in first row
we choose the second smallest cost in the matrix. It is in the cell (1,4).
$\boldsymbol{X_{14}} = \min(a_1 , b_4) = (12 , 7) = 7$, **remove fourth column** and remain only 5 in first row.
We choose the third smallest cost in the matrix. It is in the cell (2,1).
$\boldsymbol{X_{21}} = \min(a_2 , b_1) = (9 ,10) = 9$,remove second row and remaining only 1 in the first column
We choose the third smallest cost in the matrix. It is in the cell (3,3).
$\boldsymbol{X_{33}} = \min(a_3 , b_3) = (11 ,15) = 11$,remove third row and remaining only 4 in the third column .
We choose the third smallest cost in the matrix. It is in the cell (1,1).
$\boldsymbol{X_{11}} = \min(a_1 , b_1) = (1 ,5) = 1$, remove first column and remaining only 4 in the first row.
The last cost remains in the cell (1 ,3) $which\ is\ X_{13} = 4$

Centers \ Units	D_1	D_2	D_3	D_4	Total available quantities
S_1	2 $X_{11}=1$	1 $X_{12}=8$	2 $X_{13}=4$	0 $X_{14}=7$	20
S_2	1 $X_{21}=9$	2 $X_{22}=0$	3 $X_{23}=0$	0 $X_{24}=0$	9
S_3	4 $X_{31}=0$	2 $X_{32}=0$	1 $X_{33}=11$	0 $X_{34}=0$	11
Total demand quantities	10	8	15	7	40 / 40

$$Min\ Z = \sum_{j=1}^{n} \sum_{i=1}^{m} C_{ij} X_{ij}$$

$$= 1 \times 2 + 8 \times 1 + 4 \times 2 + 9 \times 1 + 11 \times 1 = 38$$

We see that the number of occupying cells are $m + n - 1 = 3 + 4 - 1 = 6$

Example 7. Find the optimal solution of the following transportation problem.

Centers→ / Units ↓	D_1	D_2	D_3	Total available quantities
S_1	8	7	2	100
S_2	4	9	10	75
S_3	2	2	8	25
S_4	5	6	11	125
Total demand quantities	150	125	130	325 / 405

Solution: $\sum a_i < \sum b_j$, that is, the given T.P. problem is **unbalanced**. We introduce a dummy source in the transportation table. The cost of transportation form this dummy source to any destination are all set equal to zero. The requirement at this dummy source is assumed to be equal to $405 - 325 = 80$.

Centers→ / Units ↓	D_1	D_2	D_3	**Total available quantities**
S_1	8	7	2	100
S_2	4	9	10	75
S_3	2	2	8	25
S_4	5	6	11	125
S_5	0	0	0	80
Total demand quantities	150	125	130	405 / 405

9. Assignment problems

This chapter deals with a very interesting method called the 'Assignment Technique' which is applicable to a class of very practical problems generally called 'Assignment problem'.

The name 'Assignment problem' originates from the classical problem where the objective is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost (or maximum profit). To examine the nature of assignment problem, **suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though will varying degree of efficiency. Let C_{ij} be the cost (payment) if the ith person is assigned the $jth\, job$, the problem is to find an assignment (which job should be assigned to which person) so that the total cost for performing all jobs is minimum.**

Further, such types of problems may consist of assigning men to offices, classes to rooms, drives to trucks, trucks to delivery routes, or problems to research teams, etc. The assignment problem can be stated in the form of $n \times n$ cost-matrix $[C_{ij}]$of real number as given in the following table.

jobs→ ↓ persons	1	2	⋯	j	⋯	n
1	C_{11}	C_{12}		C_{1j}		C_{1n}
2	C_{21}	C_{22}		C_{2j}		C_{2n}
⋮						
i	C_{i1}	C_{i2}		C_{ij}		C_{in}
⋮						
n	C_{n1}	C_{n2}		C_{nj}		C_{nn}

9.1. Mathematical Formulation of assignment problem

$$Minimize\ the\ total\ cost: Z = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij}\ ,\ i = 1, \dots, n;\ \ j = 1, \dots, n$$

$$X_{ij} = \begin{cases} 1 & if\ the\ person\ is\ assigned\ jth\ job \\ 0 & if\ not \end{cases}$$

$\boldsymbol{\sum_{j=1}^{n} X_{ij} = 1}$ (one job is done by the ith person, $i = 1, \dots, n$) and
$\sum_{i=1}^{n} X_{ij} = 1$ (only one person should be assigned the jth job, $j = 1, \dots, n$)
Where X_{ij} denotes that $jth\ job$ is to be assigned to the $ith\ person$.

This special structure of assignment problem allows a more convenient method of solution in comparison to simplex method.

9.2. Fundamental Theorems

The solution to an assignment problem is fundamentally based on the following tow theorems. .

Theorem 1. Reduction theorem: In an assignment problem, if we add (or subtract) a constant to every element of a row (or column) of the cost matrix $[C_{ij}]$, then an assignment plan that minimizes the total cost for the new cost matrix also minimizes the total cost for the original cost matrix.

Corollary. If (X_{ij}), $i = 1, \dots, n$ is an optimal solution for an assignment problem with cost (C_{ij}), then it is also optimal for the problem with cost (C'_{ij}) when
$C_{ij} = C'_{ij}$ for $i\ , j = 1\ , \dots, n\ , j \neq k$
$C'_{ik} = C_{ik} - A\ , where\ A\ is\ a\ constant.$

Theorem 2. In an assignment problem with cost $(C_{ij}), if\ C_{ij} \geq 0$ then a feasible solution (X_{ij}) which satisfying $\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij} = 0$, is optimal for the problem.

Theorem . (König Theorem). Let P be the set of 0 elements of a matrix C. Then the maximum number of 0's that can be selected in P such that no row or column of C contains more than one such 0 is equal to the minimum number of lines covering all the elements of P.

Corollary. The maximum subset of P provides an optimal assignment when the minimum number of lines to cover all the elements of P is equal to the order of C.

8.3. The Hungarian Method For Assignment problem algorithm: An example1.

We consider an example where four jobs (J1, J2, J3, and J4) need to be executed by four workers (W1, W2, W3, and W4), one job per worker. The matrix below shows the cost of assigning a certain worker to a certain job. The objective is to minimize the total cost of the assignment.

	J1	J2	J3	J4
W1	82	83	69	92
W2	77	37	49	92
W3	11	69	5	86
W4	8	9	98	23

Below we will explain the Hungarian algorithm using this example. Note that a general description of the algorithm can be found here.

Solution. Step 1: Subtract row minima

We start with subtracting the row minimum from each row. The smallest element in the first row is, for example, 69. Therefore, we subtract 69 from each element in the first row. The resulting matrix is:

	J1	*J2*	*J3*	*J4*	
W1	13	14	0	23	(-69)
W2	40	0	12	55	(-37)
W3	6	64	0	81	(-5)
W4	0	1	90	15	(-8)

Step 2: Subtract column minima

Similarly, we subtract the column minimum from each column, giving the following matrix:

	J1	*J2*	*J3*	*J4*
W1	13	14	0	8
W2	40	0	12	40
W3	6	64	0	66
W4	0	1	90	0
				(-15)

Step 3: Cover all zeros with a minimum number of lines

We will now determine the minimum number of lines (horizontal or vertical) that are required to cover all zeros in the matrix. All zeros can be covered using 3 lines:

	J1	*J2*	*J3*	*J4*
W1	13	14	[0]	8
W2	40	[0]	12	40
W3	6	64	×	66
W4	[0]	1	96	×

Because the number of lines required (3) is lower than the size of the matrix (*n*=4), we continue with Step 4.

Step 4: Create additional zeros

First, we find that the smallest uncovered number is 6. We subtract this number from all uncovered elements and add it to all elements that are covered twice. This results in the following matrix:

	J1	*J2*	*J3*	*J4*
W1	7	8	0	2
W2	40	0	18	40
W3	0	58	0	60
W4	0	1	96	0

Now we return to Step 3.

Step 3: Cover all zeros with a minimum number of lines

Again, We determine the minimum number of lines required to cover all zeros in the matrix. Now there are 4 lines required:

	J1	*J2*	*J3*	*J4*
W1	7	8	[0]	2
W2	40	[0]	18	40
W3	[0]	58	×	60
W4	×	1	96	[0]

Because the number of lines required (4) equals the size of the matrix (n=4), an optimal assignment exists among the zeros in the matrix. Therefore, the algorithm stops.

The optimal assignment

The following zeros cover an optimal assignment:

	J1	*J2*	*J3*	*J4*
W1	7	8	[0]	2
W2	40	[0]	18	40
W3	[0]	58	0	60
W4	0	1	96	[0]

This corresponds to the following optimal assignment in the original cost matrix:

	J1	J2	J3	J4
W1	82	83	69	92
W2	77	37	49	92
W3	11	69	5	86
W4	8	9	98	23

Thus, worker 1 should perform job 3, worker 2 job 2, worker 3 job 1, and worker 4 should perform job 4. The total cost of this optimal assignment is to 69 + 37 + 11 + 23 = 140.

Example 2. An account officer has 4 subordinates and 4 tasks, The subordinates differ in efficiency. The tasks also differ in their intrinsic difficulty. His estimates of the time each would take to perform each task is given in the matrix below. How should the tasks be allocated one to one man, so that the total man hours are minimized?

Task→ / man↓	**T1**	**T2**	**T3**	**T4**
1	8	26	17	11
2	13	28	4	26
3	38	19	18	15
4	19	26	24	10

Solution. Step 1: Subtract row minima

We start with subtracting the row minimum from each row. The smallest element in the first row is, 8. Therefore, we subtract 8 from each element in the first row. The resulting matrix is:

	T1	T2	T3	T4	
1	0	18	9	3	(-8)
2	9	24	0	22	(-4)
3	23	4	3	0	(-15)
4	9	16	14	0	(-10)

Step 2: Subtract column minima

Similarly, we subtract the column minimum from each column, giving the following matrix:

	T1	T2	T3	T4
1	0	14	9	3
2	9	20	0	22
3	23	0	3	0
4	9	12	14	0
		(-4)		

Step 3: Cover all zeros with a minimum number of lines

Again, We determine the minimum number of lines required to cover all zeros in the matrix. Now there are 4 lines required:

	T1	T2	T3	T4
1	[0]	14	9	3
2	9	20	[0]	22
3	23	[0]	3	×
4	9	12	14	[0]

Because the number of lines required (4) equals the size of the matrix (n=4), an optimal assignment exists among the zeros in the matrix. Therefore, the algorithm stops.

The optimal assignment

The following zeros cover an optimal assignment:

	T1	**T2**	**T3**	**T4**
1	[0]	14	9	3
2	9	20	[0]	22
3	23	[0]	3	0
4	9	12	14	[0]

$1 \to T1, 2 \to T3, 3 \to T2, 4 \to T4.$

Minimum time taken$= 8 + 4 + 19 + 10 = 41$

Homework. (1) A manager has 5 jobs to be done. The following matrix shows the time taken by the $jth\ job$, $(j = 1, 2, \ldots, 5)$ on the $ith\ machine$ $(i = 1,2, \ldots, 5)$. Assign 5 jobs to the 5 machines so that the total time taken is minimized.

Job→ / Machine ↓	**J1**	**J2**	**J3**	**J4**	**J5**
1	9	3	4	2	10
2	12	10	8	11	9
3	11	2	9	0	8
4	8	0	10	2	1
5	7	5	6	2	9

(2) Consider three jobs to be assigned to three machines. The cost for each combination is shown in the table below. Determine the minimal job-machine combinations.

→ Machine / Job↓	**A**	**B**	**C**
1	5	7	9
2	14	10	12
3	15	13	16

3. Consider four jobs to be assigned to four machines. The cost for each combination is shown in the table below. Determine the minimal job – machine combinations.

Machine→ / Job↓	**A**	**B**	**C**	**D**
1	1	4	6	3
2	8	7	10	9
3	4	5	11	9
4	6	7	8	5

8.4. Unbalanced assignment problems

Like the unbalanced transportation problems there could be arise unbalanced assignment problems too. They are to be handled exactly in the same manner , this mean that by introducing dummy jobs or dummy men, etc. The following unbalanced problem serves as an example.

Example3. Consider Unbalanced Assignment problem .

Machine→ / Job↓	**A**	**B**	**C**	**D**	**E**
1	5	7	11	6	7
2	8	5	5	6	5
3	6	7	10	7	3
4	10	4	8	2	4

Solution. Convert the $4 \times 5\ matrix$ into a square matrix by adding a dummy row D_5

Machine→ / Job↓	**A**	**B**	**C**	**D**	**E**
1	5	7	11	6	7
2	8	5	5	6	5
3	6	7	10	7	3
4	10	4	8	4	2
$\boldsymbol{D_5}$	0	0	0	0	0

Step 1: Subtract row minima

Machine→ / Job↓	**A**	**B**	**C**	**D**	**E**
1	0	2	6	1	2
2	3	0	0	1	0
3	3	4	7	4	0
4	8	2	6	2	0
$\boldsymbol{D_5}$	0	0	0	0	0

Step 2: Subtract column minima

Column-wise reduction is not necessary since all columns contain a single zero.

Step 3: Cover all zeros with a minimum number of lines

Machine→ / Job↓	**A**	**B**	**C**	**D**	**E**
1	[0]	2	6	1	2
2	3	[0]	×	1	×
3	3	4	7	4	[0]
4	8	2	6	2	×
$\boldsymbol{D_5}$	×	×	[0]	×	×

Number of lines drawn ≠Order of a matrix. Hence not optimal, we continue with step 4

Step 4: Create additional zeros

First, we find that the smallest uncovered number is 1. We subtract this number from all uncovered elements and add it to all elements that are covered twice. This results in the following matrix:

Machine→ / Job↓	**A**	**B**	**C**	**D**	**E**
1	[0]	1	5	×	2
2	4	[0]	×	1	1
3	3	3	6	3	[0]
4	8	1	5	1	×
$\boldsymbol{D_5}$	1	×	[0]	×	1

Number of lines drawn ≠Order of a matrix. Hence not optimal, we continue with step 4

Again added or subtracted 1 from elements.

Machine→ / Job↓	**A**	**B**	**C**	**D**	**E**
1	[0]	1	5	×	3
2	4	[0]	×	1	2
3	2	2	5	2	[0]
4	7	×	4	[0]	×
$\boldsymbol{D_5}$	1	×	[0]	×	2

Number of lines drawn =order of matrix. Hence optimality is reached . Now assign the jobs to machines as shown below.

$1 \rightarrow A, 2 \rightarrow B, 3 \rightarrow E, 5 \rightarrow D, D_5 \rightarrow C$

The total cost of this optimal assignment is to 5 + 5 + 3 + 2+0 = 15 $