IT 101

Content

3-12 ((Calculating The Present Value Of An Ordinary Annuity And An Annuity Due By Formula))

3-13 ((Calculating Amortization Payments By Formula))

References

1- **Cheryl Cleaves , Margie Hobbs and Jeffry Noble**

2- James Stewart , Lothar Redlin and Saleem Watson

3- Robert brechner and George Bergeman

4- Thomas Calculus

UNIT 1
SEQUENCES
AND
SERIES

A sequence is an infinite list of numbers. The numbers in the sequence are often written as a_1 , $a_{2,}$ $a_{3,}$ . . . . The dots mean that the list continues forever.

A simple example is the sequence

5 ,	10,	15,	20,	25,	. . .
↑	↑	↑	↑	↑	
$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	. . .

We can describe the pattern of the sequence displayed above by the following formula :

$\boldsymbol{a_n}$ = 5n

You go from one number to the next by adding **5** . This natural way of describing the sequence

is expressed by the recursive formula:

$a_n = \boldsymbol{a_{n-1}} + 5$

starting with $\boldsymbol{a_1}$ = 5 . Try substituting n = 1, 2, 3, . . . in each of these formulas to see how they produce the numbers in the sequence .

<u>Definition</u> (Sequence)

A sequence is a function $\boldsymbol{a}$ whose domain is the set of natural numbers. The terms of the sequence are the function values

$a(1)$, $a(2)$, $a(3)$, . . . , a(n) , . . .

We usually write an instead of the function notation a(n) . So the terms of the sequence are written as

a_1 , a_2 , a_3, . . . , a_n . . .

The number a_1 is called the first term, , a_2 is called the second term, and in general, a_n is called the nth term.

Here is a simple example of a sequence:

2, 4, 6, 8, 10, . . .

This sequence consists of even numbers . This can be done by giving a formula for the nth term a_n of the sequence. In this case ,

$\boldsymbol{a_{n=2n}}$ and the sequence can be written as

2 ,	*4 ,*	*6 ,*	*8 ,*	*. . .*	*2n ,*	*. . .*
1st term	*2nd term*	*3rd term*	*4th term*		*nth term*	

Notice how the formula $a_n = 2n$ *gives all the terms of the sequence. For instance, substituting*
1, 2, 3, *and* ***4*** *for* n *gives the first four terms:*

$a_1 = 2 . 1 = 2$ $a_2 = 2 . 2 = 4$ $a_3 = 2 . 3 = 6$ $a_4 = 2 . 4 = 8$

To find the 103rd term of this sequence, we use $n = 103$ to get

$a_{103} = 2 . 103 = 206$

Example 1 ■ Finding the Terms of a Sequence

Find the first five terms and the 100th term of the sequence defined by each formula

(a) $a_n = 2n - 1$ (b) $c_n = n^2 - 1$ (c) $t_n = \frac{n}{n+1}$ (d) $r_n = \frac{(-1)^n}{2^n}$

Solution : To find the first five terms, we substitute **n = 1, 2, 3, 4,** and 5 in the formula for the nth term. To find the 100th term, we substitute n = 100. This gives the following.

nth term	First five terms	100th term
(a) $a_n = 2n - 1$	$1\ 3, 5, 7, 9$	199
(b) $c_n = n^2 - 1$	$0, 3, 8, 15, 24$	9999
(c) $t_n = \frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$	$\frac{100}{101}$
(d) $r_n = \frac{(-1)^n}{2^n}$	$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$	$\frac{1}{2^{100}}$

Remark

In Example 1(d) the presence of $(-1)^n$ in the sequence has the effect of making successive terms alternately negative and positive.

It is often useful to picture a sequence by sketching its graph.
Since a sequence is a function whose domain
is the natural numbers, we can draw its graph in the Cartesian plane.
For instance, the graph of the sequence

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots$ is show in figure **1**

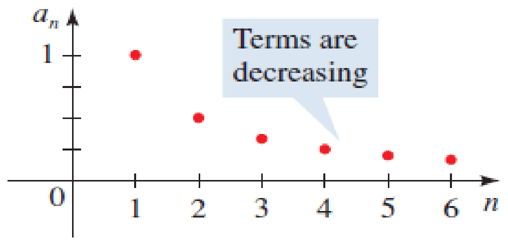


FIGURE 1

Compare the graph of the sequence shown in Figure **1** to the graph of

$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots, \frac{(-1)^n}{n}, \ldots$

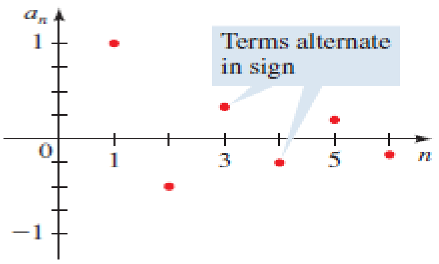


FIGURE 2

Homework :

Find the first four terms and the 100th term of the sequence whose nth term is given.

1) $a_n = n - 3$ 2) $a_n = \frac{1}{2n+1}$ 3) $a_n = 5^n$ 4) $a_n = \frac{(-1)^n}{n^2}$

Example 2 ■ Finding the nth Term of a Sequence

Find the nth term of a sequence whose first several terms are given .

a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ b) $-2, 4, -8, 16, -32, \ldots$

Solution

(**a**) We notice that the numerators of these fractions are the odd numbers and the denominators are the even numbers. Even numbers are of the form **2n**, and odd numbers are of the form

2n-1 (an odd number differs from an even number by **1**). So a sequence that has these numbers for its first four terms is given by

$a_n = \frac{2n-1}{2n}$

(**b**) These numbers are powers of 2, and they alternate in sign, so a sequence that agrees with these terms is given by

$a_n = (-1)^n 2^n$

Homework : ■ *n*th term of a Sequence

Find the *n*th term of a sequence whose first several terms are given.

1) 2 , 4 ,8 , 16 , . . . 2) $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

■ Recursively Defined Sequences

Some sequences do not have simple defining formulas like those of the preceding example.

The nth term of a sequence may depend on some or all of the terms preceding it. A sequence defined in this way is called recursive.

Example 3 ■ Finding the Terms of a Recursively Defined Sequence

A sequence is defined recursively by $a_1 = 1$ and $a_n = 3(a_{n-1} + 2)$

Find the first five terms of the sequence.

Solution :The defining formula for this sequence is recursive. It allows us to find the **nth** term a_n if we know the preceding term a_{n-1} . Thus we can find the **second term** from the **first term**, the third term from the second term, the **fourth term** from the **third term**, and so on. Since we are given the first terma_1**=1**, we can proceed as follows.

$a_2 = 3(a_1 + 2) = 3(1+2) = 9$

$a_3 = 3(a_2 + 2) = 3(9 + 2) = 33$

$a_4 = 3(a_3 + 2) = 3(33 + 2) = 105$

$a_5 = 3(a_4 + 2) = 3(105 + 2) = 321$

Thus the first five terms of this sequence are

1, 9, 33, 105, 321, . . .

Homework

A sequence is defined recursively by the given formulas. Find the first five terms of the sequence.

1) $a_n = 2(a_{n-1} + 3)$ and $a_1 = 4$

2) Find the first ten terms of the sequence$a_n = \frac{1}{a_{n-1}}$ $a_1 = 2$

Example 4 ■ **The Fibonacci Sequence**

Find the first 11 terms of the sequence defined recursively by F1 = 1, F2 = 1

$F_n = F_{n-1} + F_{n-2}$

Solution : To find F_n, we need to find the two preceding terms,F_{n-1} and F_{n-2}.
Since we are given $F_1$and F_2 , we proceed as follows.

$F_3 = F_1 + F_2 = 1 + 1 = 2$

$F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = F_4 + F_3 = 3 + 2 = 5$

It's clear what is happening here. Each term is simply the sum of the two terms that precede it, so we can easily write down as many terms as we please. Here are the first 11 terms. (You can also find these using a graphing calculator.)

1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 , 34 , 55 , 89 , . . .

Homework:

sequence is defined recursively by the given formulas. Find the first five terms of the sequence .

$a_n = a_{n-1} + a_{n-2}$ and $a_1 = 1$ $a_2 = 1$

▪ 1-2 Definition (The Partial Sums of a Sequence)

For the sequence

$a_1 , a_2 , a_3 , a_4 , . . . , a_n , . . .$

the partial sums are

$s_1 = a_1$

$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$s_4 = a_1 + a_2 + a_3 + a_4$
$\vdots$
$s_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$
$\vdots$

s_1 is called the first partial sum, s_2 is the second partial sum, and so on. s_n is called the nth partial sum. The sequence $s_1, s_2, s_3, \ldots, s_n, \ldots$ is called the sequence of partial sums.

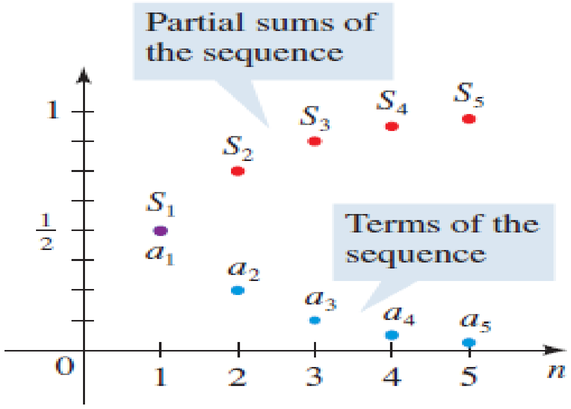


FIGURE 7 Graph of the sequence a_n and the sequence of partial sums S_n

Example 5 ■ Finding the Partial Sums of a Sequence

Find the first four partial sums , 10th term and the nth partial sum of the sequence given by

$\boldsymbol{a_n} = \frac{1}{2^n}$

Solution The terms of the sequence are

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

The first four partial sums are

$s_1 = a_1 = \frac{1}{2} \qquad = \frac{1}{2}$

$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} \qquad = \frac{3}{4}$

$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = \frac{7}{8}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

Notice that in the value of each partial sum, the denominator is a power of 2 and the numerator is one less than the denominator. In general, the *n*th partial sum is

$s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$

Example 6 ■ Finding the Partial Sums of a Sequence

Find the first four partial sums and the *n*th partial sum of the sequence given by

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Solution The first four partial sums are

$s_1 = 1 - \frac{1}{2} \qquad = 1 - \frac{1}{2}$

$s_2 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) \qquad = 1 - \frac{1}{3}$

$s_3 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) \qquad = 1 - \frac{1}{4}$

$s_4 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) \quad = 1 - \frac{1}{5}$

The *n*th partial sum is

$$s_n = 1 - \frac{1}{n+1}$$

Homework

Find the first four partial sums and the *n*th partial sum of the sequence a_n .

1) $a_n = \frac{2}{3^n}$

2) $a_n = \sqrt{n} - \sqrt{n+1}$

Definition ■ (Sigma Notation)

Given a sequence $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$, $\boldsymbol{a_4}$, . . .

we can write the sum of the first n terms using **summation notation**, or **sigma notation**. Sigma notation is used as follows:

$$\sum_{k=1}^{n} \boldsymbol{a_k} = \boldsymbol{a_1} + \boldsymbol{a_2} + \boldsymbol{a_3} + \boldsymbol{a_4} + \ldots + \boldsymbol{a_n}$$

The left side of this expression is read, "The sum of $\boldsymbol{a_k}$ from **k = 1** to **k = n**." The letter **k**is called the index of summation, or the summation variable, and the idea is to replace **k**in the expression after the sigma by the integers **1, 2, 3, . . . , n,** and add the resulting expressions, arriving at the right-hand side of the equation.

Example 7 ■ Sigma Notation

Find each sum.

a) $\sum_{k=1}^{5} k^2$ b) $\sum_{j=3}^{5} \frac{1}{j}$ c) $\sum_{k=5}^{10} k$ d) $\sum_{i=1}^{6} 2$

Solution

$(a) \sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

b) $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

c) $\sum_{k=5}^{10} k = 5 + 6 + 7 + 8 + 9 + 10 = 45$

d) $\sum_{i=1}^{6} 2 = 2 + 2 + 2 + 2 + 2 + 2 = 12$

Homework

Find the sum.

1) $\sum_{k=1}^{4} k$ 2) $\sum_{k=1}^{3} \frac{1}{k}$

Example 8: Write each sum using sigma notation .

a) $1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$

b) $\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77}$

Solution

(a) We can write

$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = \sum_{k=1}^{7} k^3$

(b) A natural way to write this sum is

$\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77} = \sum_{k=3}^{77} \sqrt{k}$

However, there is no unique way of writing a sum in sigma notation. We could also write this sum as

$\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77} = \sum_{k=1}^{75} \sqrt{k+2}$

Or $\sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots + \sqrt{77} = \sum_{k=0}^{75} \sqrt{k+3}$

<u>Homework</u> Write the sum using sigma notation .

1) $2 + 4 + 6 + \ldots + 50$
2) $1^2 + 2^2 + 3^2 + \ldots + 10^2$

Property Of Sums

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ Be sequences . Then for every positive integer n and any real number c the following properties hold.

1) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} (a_k) + \sum_{k=1}^{n} (b_k)$

2) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} (a_k) - \sum_{k=1}^{n} (b_k)$

3) $\sum_{k=1}^{n} c\, a_k = c(\sum_{k=1}^{n} a_k)$

Proof : To prove Property 1, we write out the left side of the equation to get

$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n)$

Because addition is commutative and associative, we can rearrange the terms on the right-hand side to read

$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + a_2 + a_3 + a_4 + \ldots + a_n) +$

$(b_1 + b_2 + b_3 + b_4 + \ldots + b_n)$

$= \sum_{k=1}^{n} (a_k) + \sum_{k=1}^{n} (b_k)$

Rewriting the right side using sigma notation gives Property 1. Property 2 is proved in a similar manner. To prove Property 3, we use the Distributive Property:

$\sum_{k=1}^{n} c\, a_k = c\, a_1 + c\, a_2 + c a_3 + c\, a_4 + \ldots + c a_n)$

$= c\,(a_1 + a_2 + a_3 + a_4 + \ldots + a_n)$

$= c(\sum_{k=1}^{n} a_k)$

2-2 Arithmetic Sequences

Perhaps the simplest way to generate a sequence is to start with a number $\boldsymbol{a}$ and add to it a fixed constant **d**, over and over again.

<u>1-3 Definition</u> (Arithmetic Sequences)

An arithmetic sequence is a sequence of the form

a, a+ d, a+ 2d, a + 3d, a+ 4d, . . .

The number $\boldsymbol{a}$ is the first term, and **d** is the common difference of the sequence. The nth term of an arithmetic sequence is given by

$a_n = a + (n-1)\, d$

<u>Remark :</u>

The number **d** is called the common difference because any two consecutive terms of an arithmetic sequence differ by **d**.

<u>Example 1</u> ■ Arithmetic Sequences

(a) If $\boldsymbol{a}$= **2** and **d** = **3**, then we have the arithmetic sequence

2, 2 + 3, 2 + 6, 2 + 9, . . .   or   2, 5, 8, 11, . . .

Any two consecutive terms of this sequence differ by **d** = **3**. The nth term is

$\boldsymbol{a_n} = \mathbf{2 + 3(n - 1)}$

(b) Consider the arithmetic sequence

9, 4, − 1, -6, − 11, . . .

Here the common difference is d = − 5. The terms of an arithmetic sequence decrease if the common difference is negative. The nth term is $\boldsymbol{a_n} = \mathbf{9 - 5(n - 1)}$

(c) The graph of the arithmetic sequence $\boldsymbol{a_n} = \mathbf{1 + 2(n - 1)}$

is shown in **Figure 1**. Notice that the points in the

graph lie on the straight line y = 2x − 1, which has slope d = 2

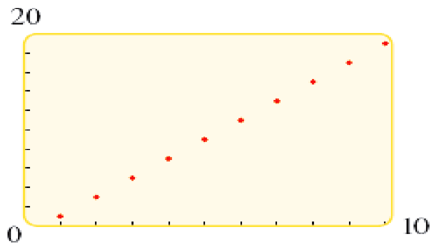


FIGURE 1

<u>Remark:</u>

An arithmetic sequence is determined completely by the first term **a** and the common difference **d**. Thus if we know the first two terms of an arithmetic sequence, then we can find a formula for the nth term, as the next example shows.

<u>Example 2</u> ■ Finding Terms of an Arithmetic Sequence

Find the common difference, the first six terms, the nth term, and the 300th term of the arithmetic sequence

13, 7, 1, − 5, . . .

<u>Solution</u> Since the first term is **13**, we have **a** = **13**. The common difference is **d = 7−13 = − 6** . Thus the **nth** term of this sequence is

$a_n = 13 - 6(n - 1)$

From this we find the first six terms:

13, 7, 1, − 5, − 11, − 17, . . .

The 300th term is

$a_{300} = 13 - 6(300 - 1) = -1781$

<u>Homework</u>

1) The **nth** term of an arithmetic sequence is given. (a) Find the first five terms ofthe sequence. (b) What is the common difference **d** ?

$a_n = 7 + 3(n - 1)$

2) Find the **nth**term of the arithmetic sequence with given first term **a** and common difference **d** . What is the 10th term?

a = 9 , d = 4

3) The first four terms of a sequence are given. Can these terms be the terms of an arithmetic

sequence? If so, find the common difference .

11 , 17 , 23 , 29 , . . .

4) Determine the common difference, the fifth term, the **nth** term, and the **100th** term of the arithmetic sequence.

4 , 10 , 16 , 22 , . . .

Example 3 ■ Finding Terms of an Arithmetic Sequence

The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find the 1000th term. Solution To find the nth term of this sequence, we need to find a and d in the formula

$a_n = a + (n - 1)d$

From this formula we get

$a_{11} = a + (11 - 1)d = a + 10d$

$a_{19} = a + (19 - 1)d = a + 18d$

Since $a_{11} = 52$ and $a_{19} = 92$ we get the following tow equations :

$$\begin{cases} 52 = a + 10d \\ 92 = a + 18d \end{cases}$$

Solving this system for a and d, we get $a = 2$ and d = 5. (Verify this.) Thus the **nth** term of this sequence is

$a_n = 2 + 5(n - 1)$

The 1000th term is $\boldsymbol{a_{1000} = 2 + 5(1000 - 1) = 4997}$

<u>Homework :</u>

1) The fourteenth term is $\frac{2}{3}$and the ninth term is $\frac{1}{4}$Find the first term and the nth term.

■(1- 4) Partial Sums of Arithmetic Sequences (Gauss method)

Suppose we want to find the sum of the numbers 1, 2, 3, 4, . . . , 100, that is,

$$\sum_{k=1}^{100} k$$

Gauss idea was this : Since we are adding numbers produced according to a fixed pattern, there must also be a pattern (or formula) for finding the sum. we started by writing the numbers from 1 to 100 and then below them wrote the same numbers in reverse order. Writing **S** for the sum and adding corresponding terms give

s = 1 + 2 + 3 + 4 + . . . + 97 + 98 + 99 + 100

s = 100 + 99 + 98 + 97 + . . . + 4 + 3 + 2 + 1

$2s = 101 + 101 + 101 + 101 + . . . + 101 + 101 + 101 + 101$

It follows that $2s = 100\,(101) = 10100$ so $s = 5050$ **.**

We want to find the sum of the first n terms of the arithmetic sequence whose terms are $a_k = a + (k - 1)\,d$, that is we want to find

$$s_n = \sum_{k=1}^{n} [\boldsymbol{a} + (\boldsymbol{k} - \mathbf{1})\boldsymbol{d}]$$

$$= a + (a + d) + (a + 2d) + (a + 3d) + . . . + [a + (n - 1)d]$$

Using Gauss's method, we write

$$s_n = \quad a \quad + \quad (a + d) \quad + \; . . . \; + [a + (n - 2)d] + [a + (n - 1)d]$$

$$s_n = [a + (n - 1)d] + [a + (n - 2)d] + \; . . . \; + \quad (a + d) \quad + \quad a$$

$$2s = [2a + (n - 1)d] + [2a + (n - 1)d + \; . . . \; + [2a + (n - 1)d + [2a + (n - 1)d]$$

$$2\,s_n = n\,[2a + (n - 1)d]$$

$$s_n = \frac{n}{2}\,[2a + (n - 1)d]$$

Notice that $a_n = a + (n - 1)\,d$ is the *n*th term of this sequence. So we can write

$$\boldsymbol{s_n} = \frac{\boldsymbol{n}}{\mathbf{2}}\,[\boldsymbol{a} + \boldsymbol{a} + (\mathbf{n} - \mathbf{1})\mathbf{d}] = \mathbf{n}\left(\frac{\boldsymbol{a} + \boldsymbol{a_n}}{\mathbf{2}}\right)$$

This last formula says that the sum of the first n terms of an arithmetic sequence is the average of the first and nth terms multiplied by n, the number of terms in the sum. We now summarize this result.

<u>Definition</u> (Partial Sums of an Arithmetic Sequence)

For the arithmetic sequence given by $\boldsymbol{a_n} = \mathbf{a + (n-1)d}$ the nth partial sum

$\boldsymbol{s_n} = a + (a + d) + (a + 2d) + (a + 3d) + . . . + [a + (n - 1)d]$

is given by either of the following formulas.

1) $\boldsymbol{s_n} = = \frac{\boldsymbol{n}}{\mathbf{2}}[\mathbf{2a + (n - 1)d}]$ **2)** $\boldsymbol{s_n} = \mathbf{n}\left(\frac{\boldsymbol{a} + \boldsymbol{a_n}}{\mathbf{2}}\right)$

Example 4 ■ Finding a Partial Sum of an Arithmetic Sequence

Find the sum of the first **50** odd numbers.

<u>Solution</u> The odd numbers form an arithmetic sequence with $\boldsymbol{a} = \mathbf{1}$ and $\mathbf{d = 2}$. The nth term is

$\boldsymbol{a_n} = 1 + 2(n - 1) = 2n - 1$, so the **50th** odd number is

$\boldsymbol{a_{50}} = 2(50-1) = 99$

Substituting in Formula 2 for the partial sum of an arithmetic sequence, we get

$$\boldsymbol{a_{50}} = 50\left(\frac{a + a_{50}}{2}\right) = 50\left(\frac{1+99}{2}\right) = 50 \times 50 = 2500$$

<u>Homework:</u>

1) Find the partial sums $\boldsymbol{s_n}$ of the arithmetic sequence that satisfies the given conditions.

$a = 3$, $d = 5$, $n = 20$

Example 5 ■ Finding a Partial Sum of an Arithmetic Sequence

Find the following partial sum of an arithmetic sequence:

3 + 7 + 11 + 15 + . . . + 159

Solution : For this sequence $a = 3$ and $d = 4$ so $\boldsymbol{a_n}$=3+ 4(n – 1)

To find which term of the sequence is the last term **159**, we use the formula for the **nth** term

and solve for **n**.

159 = 3 + 4(n– 1) set $\boldsymbol{a_n}$ = 159

39 = n-1 subtract 3 ; divide by 4

N = 40 add 1

To find the partial sum of the first 40 terms, we use Formula 1 for the nth partial sum of an arithmetic sequence:

$s_{40} == \frac{40}{2}$ [2(3) + 4 (40 – 1)] = 3240

Homework

A partial sum of an arithmetic sequence is given. Find the sum.

1) 1 + 5 + 9 + . . . + 401

Example 6: if $\boldsymbol{a_1}$, $\boldsymbol{a_2, a_3 ..., a_n}$are in arithmetic sequence , where $\boldsymbol{a_i > 0}$ for all **i** , show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution : **L . H .S.** $= \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$$= \frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1} + \frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2} + \ldots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

Let ‘ **d** ‘ is the common difference of this arithmetic sequence then

$\boldsymbol{a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1}}$ =d

Now L . H . S .

$$= \frac{1}{d} \left(\sqrt{a_2} - \sqrt{a_1} + \sqrt{a_3} - \sqrt{a_2} + \ldots + \sqrt{a_{n-1}} - \sqrt{a_{n-2}} + \sqrt{a_n} - \sqrt{a_{n-1}}\right)$$

$$= \frac{1}{d} \{\sqrt{a_n} - \sqrt{a_1}\}$$

$$= \frac{a_n - a_1}{d(\sqrt{a_n} + \sqrt{a_1})}$$

$$= \frac{a_1 + (n-1)d - a_1}{d(\sqrt{a_n} + \sqrt{a_1})}$$

$= \frac{1}{d}\frac{(\boldsymbol{n}-\mathbf{1})\boldsymbol{d}}{(\sqrt{\boldsymbol{a_n}}+\sqrt{\boldsymbol{a_1}})}$

$= \frac{(\boldsymbol{n}-\mathbf{1})}{\sqrt{\boldsymbol{a_n}}+\sqrt{\boldsymbol{a_1}}}$ =**R . H . S.**

Example 7 : The first, second and the last terms of an arithmetic sequence are a, b, c, respectively

Prove that the sum is

$$\frac{(\boldsymbol{a}+\boldsymbol{c})(\boldsymbol{b}+\boldsymbol{c}-\mathbf{2}\boldsymbol{a})}{\mathbf{2}(\boldsymbol{b}-\boldsymbol{a})}$$

Solution : Here first term = *a* $\therefore \boldsymbol{T_1} = a$

Second term = b, , $\therefore \boldsymbol{T_2} = b$

The common difference $d = \boldsymbol{T_2} - \boldsymbol{T_1} = \boldsymbol{b} - a$

Again last term $= \boldsymbol{T_n}$

$c = a + (n - 1)\, d$

$\Leftrightarrow n = \frac{c-a+d}{d}$

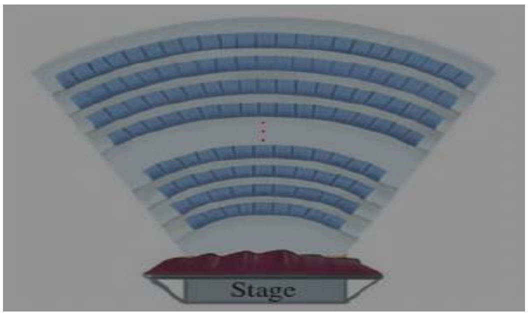
$\Leftrightarrow n = \frac{(b+c-2a)}{(b-a)}$ $(d = b - a)$

$\therefore$ *Sum of n terms* $\boldsymbol{s_n} = \frac{n}{2}(a + \boldsymbol{a_n})$

$$= \frac{(b+c-2a)(a+c)}{2(b-a)}$$

Example 6 ■ Finding the Seating Capacity of an Amphitheater

An amphitheater has **50** rows of seats with **30** seats in the first row, **32** in the second, 34 in the third, and so on. Find the total number of seats.



Solution : The numbers of seats in the rows form an arithmetic sequence with **a = 30** and **d = 2**. Since there are **50** rows, the total number of seats is the sum

$s_n = \frac{n}{2}$ [2a + (n – 1)d

$s_{50} = \frac{50}{2}$ [2(30) + 49(2) = 3950

Thus the amphitheater has 3950 seats.

Homework

Theater Seating An architect designs a theater with **15**seats in the first row, **18** in the second, **21** in the third, and so on. If the theater is to have a seating capacity of **870**, how many rows must the architect use in his design ?

Example 7 ■ Finding the Number of Terms in a Partial Sum

How many terms of the arithmetic sequence 5, 7, 9, . . . must be added to get 572 ?

Solution : We are asked to find n when s_n **= 572**. Substituting **a = 5 , d = 2 ,** and s_n **= 572** in Formula 1 for the partial sum of an arithmetic sequence, we get

$s_n = \frac{n}{2}$ [2a + (n – 1)d

572 $= \frac{n}{2}$ [2(5) + (n – 1)2

572 = 5n + n(n – 1) Distributive property

0 = n^2 + 4n – 572 Expand

0 = (n – 22) (n + 26) Factor

This gives n = 22 or n = - 26 . But since **n** is the number of terms in this partial sum , we must have **n = 22** .

Homework :

1) Adding Terms of an Arithmetic Sequence Find the number of terms of the arithmetic sequence with the given description that must be added to get a value of **2700** .

The first term is **5** , and the common difference is **2** .

1.5 Geometric Sequences

In this section we study geometric sequences. This type of sequence occurs frequently in applications to finance, population growth, and other fields.

■ Geometric Sequences

Recall that an arithmetic sequence is generated when we repeatedly add $\boldsymbol{a}$ number **d**to an initial term $\boldsymbol{a}$. A geometric sequence is generated when we start with a number $\boldsymbol{a}$ and repeatedly multiply by a fixed nonzero constant **r** .

Definition of Geometric Sequence

A geometric sequence is a sequence of the form

$\boldsymbol{a}$ **,** $\boldsymbol{a}$**r ,** $\boldsymbol{ar^2}$ **,** $\boldsymbol{ar^3}$ **,** $\boldsymbol{ar^4}$ **, . . .**

The number $\boldsymbol{a}$ **is the first term , and r is the common ratio of the sequence .The nthterm of a geometric sequence is given by**

$\boldsymbol{a_n = ar^{n-1}}$

Remark : The number **r** is called the common ratio because the ratio of any two consecutive

terms of the sequence is **r** .

Example 1■ Geometric Sequences

(a) If $\boldsymbol{a}$**= 3** and **r = 2** , then we have the geometric sequence

$$3 , 3 . 2 , 3 . 2^2 , 3 . 2^3 , 3 . 2^4 , \ . . .$$

or 3, 6, 12, 24, 48, . . .

Notice that the ratio of any two consecutive terms is **r = 2** . The **nth** term is

$a_n = 3\,(2)^{n-1}$

(b) The sequence 2 , - 10 , 50 , - 250 , 1250 , . . .

is a geometric sequence with **a = 2** and **r = -5** . When **r**is negative, the terms ofthe sequence alternate in sign. The ***n*th** term is

$a_n = 2\,(-5)^{n-1}$

(c) The sequence $1 , \frac{1}{3} , \frac{1}{9} , \frac{1}{27} , \frac{1}{81} , . . .$

is a geometric sequence with ***a*** **=1** and ***r*** = $\frac{1}{3}$. The *n*th term is

$a_n = 1\,(\frac{1}{3})^{n-1}$

(d) The graph of the geometric sequence defined by $a_n = \frac{1}{5}\,(2)^{n-1}$

is shown in Figure 1 . Notice that the points in the graph lie

on the graph of the exponential function $y = \frac{1}{5}\,(2)^{x-1}$

If $0 < r < 1$, then the terms of the geometric sequence $a(r)^{n-1}$ decrease , but if $r > 1$, then the terms increase .

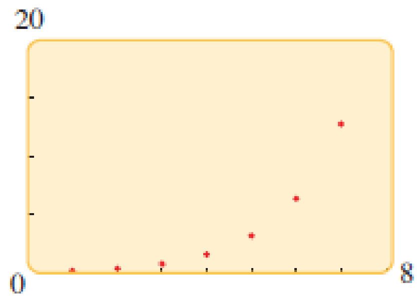


FIGURE 1

Remark :

Geometric sequences occur naturally. Here is a simple example. Suppose a ball has elasticity such that when it is dropped, it bounces up one-third of the distance it has fallen. If this ball is dropped from a height of **2 m** , then it bounces up to a height of $\mathbf{2(\frac{1}{3})m}$. On its second bounce, it returns to a height of $(\frac{2}{3})\,(\frac{1}{3}) = (\frac{2}{9})\mathbf{m}$, and so on (see Figure 2). Thus the height $h_n$that the ball reaches on its **nth** bounce is given by the geometric sequence

$h_n = \frac{2}{3}\,(\frac{1}{3})^{n-1} = 2\,(\frac{1}{3})^n$

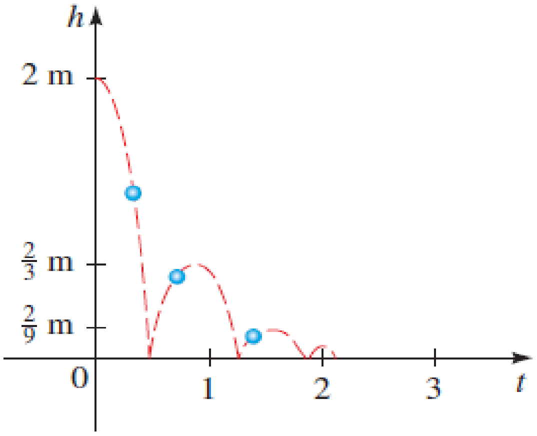


FIGURE 2

Example 2 ■ Finding Terms of a Geometric Sequence

Find the common ratio, the first term, the nth term, and the eighth term of the geometric sequence

5, 15, 45, 135, . . .

Solution To find a formula for the **nth** term of this sequence, we need to find the first term a and the common ratio **r** . Clearly, **a = 5** . To find **r** , we find the ratio of any two consecutive terms. For instance , r = $\frac{45}{15}$ = 3 . Thus

$a_n = 5\,(3)^{n-1}$ $a_n = a\,(r)^{n-1}$

The eighth term is $a_8 = 5\ (3)^{8-1} = 5(\,(3)^7 = 10935$

Remark :

The above example refers to we can find the nth term of a geometric sequence if we know any two terms.

Homework

1) The nth term of a sequence is given. **(a)** Find the first five terms of the sequence. (b) What is the common ratio **r** ?

a) $a_n = 7\,(3)^{n-1}$ b) $a_n = (3)^{n-1}$

2) Find the **nth** term of the geometric sequence with given first term a and common ratio **r** . What is the fourth term**?** **a = 7 , r = 4**

3) The first four terms of a sequence are given. Determine whether these terms can be the

terms of a geometric sequence. If the sequence is geometric, find the common ratio.

3 , 6 , 12 , 24 , . . .

4) Determine the common ratio, the fifth term, and the nth term of the geometric sequence .

2, 6, 18, 54, . . .

Example 3 ■ Finding Terms of a Geometric Sequence

The third term of a geometric sequence is $\frac{63}{4}$, and the sixth term is $\frac{1701}{32}$. Find the fifth term .

Solution Since this sequence is geometric, its nth term is given by the formula

$a_n = a\,(r)^{n-1}$. Thus

$a_3 = a\,(r)^{3-1} = a\,r^2$

$a_6 = a\,(r)^{6-1} = a\,r^5$

From the values we are given for these two terms, we get the following system of equations :

$$\begin{cases} \frac{63}{4} = a\,r^2 \\ \frac{1701}{32} = a\,r^5 \end{cases}$$

We solve this system by dividing .

$\frac{a\,r^5}{a\,r^2} = \frac{\frac{1701}{32}}{\frac{63}{4}}$, $r^3 = \frac{27}{8}$ simplify $r = \frac{3}{2}$ Take cube root of each side

Substituting for r in the first equation gives

$\frac{63}{4} = a\,(\frac{3}{2})^2$ Substitute $r = \frac{3}{2}$ in $\frac{63}{4} = a\,r^2$

$a = 7$ solve for a

It follows that the *n*th term of this sequence is

$a_n = 7\,(\frac{3}{2})^{n-1}$

Thus the fifth term is $a_5 = 7\,(\frac{3}{2})^{5-1} = 7\,(\frac{3}{2})^4 = \frac{567}{16}$

Homework :

The third term is $-\frac{1}{3}$ and the sixth term is 9. Find the first ,second and the nth terms .

1 -6■ Partial Sums of Geometric Sequences

For the geometric sequence **a , ar,** ar^2, ar^3, ar^4 , . . . , ar^{n-1} , . . . , the **nth** partial sum is

$s_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ar^4 + \ ...\ + ar^{n-1}$

To find a formula for s_n, we multiply s_n by **r** and subtract from s_n.

$s_n = a + ar + ar^2 + ar^3 + ar^4 + \ ...\ + ar^{n-1}$

$rs_n = ar + ar^2 + ar^3 + ar^4 + \ ...\ + ar^{n-1} + ar^n$

$s_n - r\,s_n = a - a\,r^n$

$s_n(1 - r) = a(1 - r^n)$

$s_n = \frac{a(1 - r^n)}{(1 - r)}$

We summarize this result by the following definition.

<u>Definition :</u> Partial Sums Of A Geometric Sequence

For the geometric sequence defined by $\boldsymbol{a_n}$= **a** $(\boldsymbol{r})^{\boldsymbol{n-1}}$ **.** The nth partial sum

$\boldsymbol{s_n}$ **= a + ar +** $\mathbf{ar^2 + ar^3 + ar^4 + \ ... \ + ar^{n-1}}$ **r ≠ 1** is given by

$$\boldsymbol{s_n} = \frac{\mathbf{a(1 - r^n)}}{\mathbf{(1 - r)}}$$

Example 4 ■ Finding a Partial Sum of a Geometric Sequence

Find the following partial sum of a geometric sequence:

1 + 4 + 16 + . . . + 4069

<u>Solution</u> For this sequence a = 1 and r = 4 , so $a_n = 4^{n-1}$. Since 4^6 = 4096 , then $4^6 = 4^{n-1}$ this leads to $6 = n - 1,\ n = 7.$ We use the formula for $s_n$with n = 7, and we have

$$s_n = 1\,\frac{(1 - 4^7)}{(1 - 4)} = 5461$$

Thus this partial sum is 5461 .

<u>Homework</u>

1)Find the partial sum $s_n$of the geometric sequence that satisfies the givenconditions .

a= 5 , r = 2 , n = 6

2) Find the sum 1 + 3 + 9 + . . . + 2187

<u>Example 5</u>■ Finding a Partial Sum of a Geometric Sequence

Find the sum $\sum_{k=1}^{6} 7(-\frac{2}{3})^{k-1}$

<u>Solution</u> The given sum is the sixth partial sum of a geometric sequence with first term

$7(-\frac{2}{3})^0 = 7$ and r = $-\frac{2}{3}$. Thus by the formula for s_n with **n = 6** we have

$$s_n = 7 \,.\, \frac{1 - (-\frac{2}{3})^6}{1 - (-\frac{2}{3})} = 7 \,.\, \frac{1 - \frac{64}{729}}{\frac{5}{3}} = \frac{931}{243} \approx 3\,.\,83$$

<u>Homework</u>: Find the sum $\sum_{k=1}^{5} 3(\frac{1}{2})^{k-1}$

1-7 Infinite Series

<u>Definition:</u> (infinite Series)

An expression of the form $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$ is called an infinite series

<u>Remark</u> : The dots mean that we are to continue the addition indefinitely. What meaning can we attach to the sum of infinitely many numbers? It seems at first that it is not possible to add infinitely many numbers and arrive at a finite number. But consider the following problem. You have a cake, and you want to eat it by first eating half the cake, then eating half of what remains, then again eating half of what remains. This process can continue indefinitely because at each stage, some of the cake remains. (See Figure 3.)

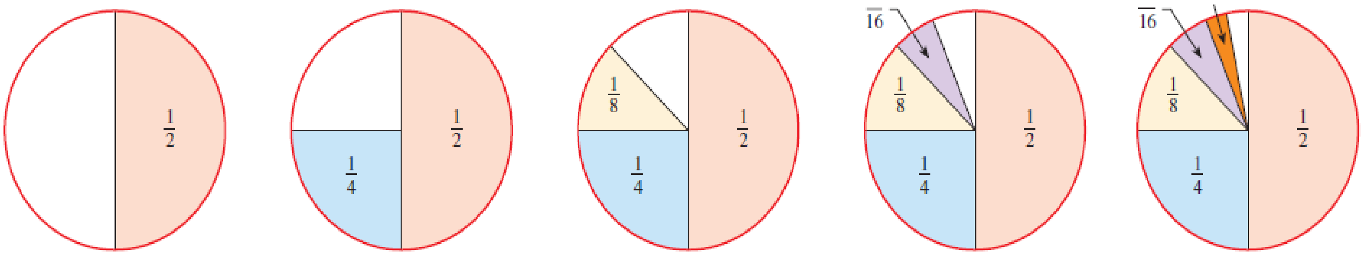


FIGURE 3

Does this mean that it's impossible to eat all of the cake? Of course not. Let's write down what you have eaten from this cake :

$$\sum_{k=1}^{\infty}\left(\frac{1}{2^k}\right) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

This is an infinite series, and we note two things about it: First, from Figure 3 it's clear that no matter how many terms of this series we add , the total will never exceed **1**. Second, the more terms of this series we add , the closer the sum is to **1** (see Figure **3**). This suggests that the number **1** can be written as the sum of infinitely many smaller numbers :

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots$$

To make this more precise, let's look at the partial sums of this series :

$$s_1 = \frac{1}{2} \qquad = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} \qquad = \frac{3}{4}$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad = \frac{7}{8}$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \qquad = \frac{15}{16}$$

and, in general ,

$$s_n = 1 - \frac{1}{2^n}$$

As **n** gets larger and larger, we are adding more and more of the terms of this series. Intuitively, as **n** gets larger s_n, gets closer to the sum of the series. Now notice that as **n** gets large, $\frac{1}{2^n}$, gets closer and closer to **0** . Thus s_n gets close to **1 - 0 = 1** . we can write

$$s_n \to 1 \qquad \text{as} \qquad n \to \infty$$

In general , if s_n gets close to a finite number **S** as **n** gets large, we say that the infinite series converges (or is convergent). The number **S** is called the sum of the infinite series. If an infinite series does not converge, we say that the series diverges (or is divergent).

Definition ■ (Infinite Geometric Series)

An **infinite geometric series** is a series of the form

a + ar + $ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + \ldots$

We can apply the reasoning used earlier to find the sum of an infinite geometric series. The **nth** partial sum of such a series is given by the formula

$$s_n = \frac{a(1 - r^n)}{(1 - r)} \qquad r \neq 1$$

It can be shown that if $|r| < 1$, then r^n gets close to **0** as n gets large (you can easily convince yourself of this using a calculator). It follows that s_n gets close to $\frac{a}{(1-r)}$ as n gets large, or

$$s_n \rightarrow \frac{a}{(1-r)} \qquad \text{as} \quad n \rightarrow \infty$$

Thus the sum of this infinite geometric series is $\frac{a}{(1-r)}$

Definition (Sum of an Infinite Geometric Series)

if $|r| < 1$ then the infinite geometric series

$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ar^4 + \ldots$

converges and has the sum

$\frac{a}{(1-r)}$ if $|r| \geq 1$, the series div

Example6 ■ Infinité Series

Determine whether the infinite geometric series is convergent or divergent. If it is convergent ,

find its sum .

(a) $2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \ldots$ B) $1 + (\frac{7}{5}) + (\frac{7}{5})^2 + (\frac{7}{5})^3 + \ldots$

Solution

(a) This is an infinite geometric series with a=2 and r = $\frac{1}{5}$. Since $|r| = \left|\frac{1}{5}\right| < 1$, the series converges. By the formula for the sum of an infinite geometric series we have

$$S = \frac{2}{(1 - \frac{1}{5})} = \frac{5}{2}$$

(b) This is an infinite geometric series with a = 1 and r = $\frac{7}{5}$. Since $|r| = \left|\frac{7}{5}\right| > 1$, the series diverges.

Homework :

Determine whether the infinite geometric series is convergent or divergent. If it is convergent ,

Find its sum

a) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ **b)** $1 + (\frac{3}{2}) + (\frac{3}{2})^2 + (\frac{3}{2})^3 + \ldots$ (C) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

(d) $\frac{1}{\sqrt{2}}+\frac{1}{2}+\frac{1}{2\sqrt{2}}+\frac{1}{4}+\ldots$ **(e)** $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots$

Example 7 ■ Writing a Repeated Decimal as a Fraction

Find the fraction that represents the rational number $2.3\underline{51}$.

Solution This repeating decimal can be written as a series:

$$\frac{23}{10}+\frac{51}{1000}+\frac{51}{100{,}000}+\frac{51}{10{,}000{,}000}+\frac{51}{1{,}000{,}000{,}000}\ \ldots$$

After the first term, the terms of this series form an infinite geometric series with

$a=\frac{51}{1000}$ and $r=\frac{1}{100}$

Thus the sum of this part of the series is

$$S=\frac{\frac{51}{1000}}{(1-\frac{1}{100})}=\frac{\frac{51}{1000}}{(\frac{99}{100})}=\frac{51}{1000}\cdot\frac{100}{99}==\frac{51}{990}$$

$$2.3\underline{51}=\frac{23}{10}+\frac{51}{990}=\frac{2328}{990}=\frac{388}{165}$$

Homework Express the repeating decimal as a fraction

(1) $0.2\underline{53}$(2)0.0303030…….. (3) $2.11\underline{25}$