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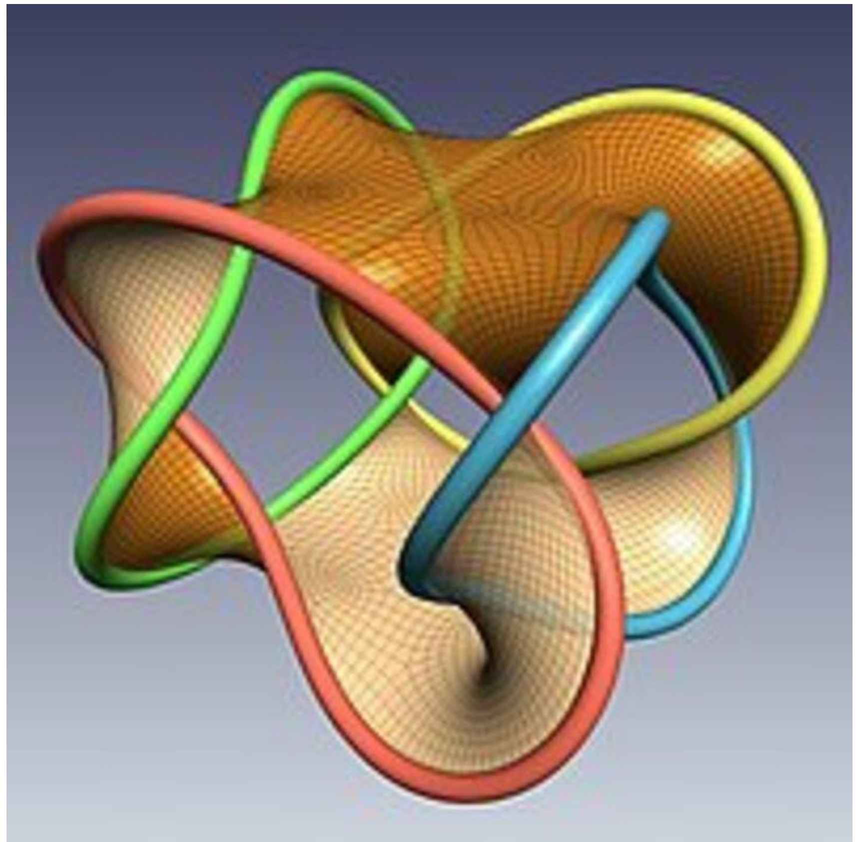
Discrete Structures



Discrete Structures:

It is the part of mathematics devoted to the stud of discrete objects (unconnected) elements.

There are several important reasons for studying discrete mathematics:

1. We can develop our mathematical ability

2. Discrete mathematic is the gateway to more advanced courses in all part of math.

3. Discrete mathematics provides the math foundations for many computer science courses

4. Discrete mathematics contains the necessary math back ground for solving problems in operation research, chemistry, engineering….

Topics: Sets, Types of set, Operations on sets.

- Sets

The word set is used in mathematics to mean any well-defined collection of items. The items in a set are called the elements of the set. For example, we can refer to the set of all the employees of a particular Company or the set of all the integers that are divisible by 5.

A specific set can be defined in two ways:

 If there are only a few elements, they can be listed individually, by writing them between braces („curly‟ brackets) and placing commas in between. For example, the set of positive odd numbers less than 10 can be written in the following way:

{1, 3, 5, 7, 9}

If there is a clear pattern to the elements, an ellipsis (three dots) can be used. For example, the set of odd numbers between 0 and 50 can be written:

{1, 3, 5, 7, ..., 49}

Some infinite sets can also be written in this way; for example, the set of all positive odd numbers can be written:

{1, 3, 5, 7, ...}

Natural numbers (N): The counting numbers {1, 2, 3, ...}.

Integers (Z): Positive and negative counting numbers, as well as zero:

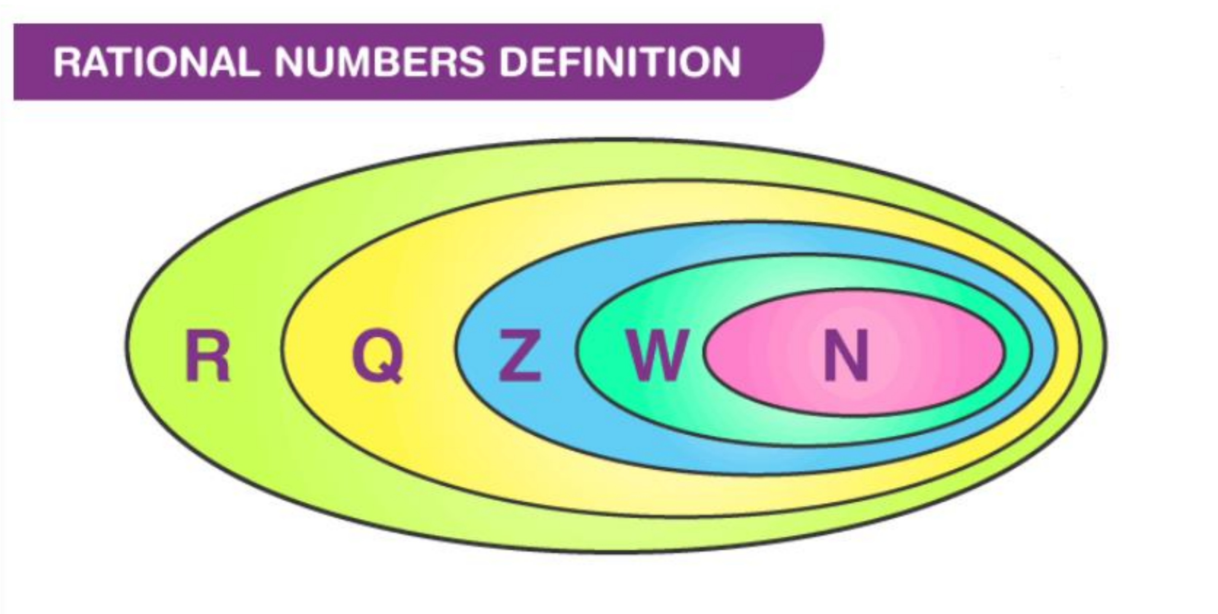
{..., −3, −2, −1, 0, 1, 2, 3, ...}.

Rational numbers (Q): Numbers that can be expressed as a ratio of an integer to a non-zero integer.

Examples of Rational Numbers:

p	q	p/q	Rational
10	2	10/2 =5	Rational
1	1000	1/1000 = 0.001	Rational
50	10	50/10 = 5	Rational

Real numbers (R): Numbers that correspond to points along a line. They can be positive, negative, or zero. All rational numbers are real, but the converse is not true.



Types of set

Subsets

Definition// Let A and B be sets. We say that B is a ***subset*** of A, and write $B \subseteq A$, if every element of B is an element of A.

For example, let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 4\}$, and $C = \{2, 4, 6\}$. Then $B \subseteq A$, but C is not a subset of A, because $6 \subseteq C$ but $6 \not\subset A$.

Definition// Two sets A and B are ***equal*** if $A \subseteq B$ and $B \subseteq A$.

In other words, $A = B$ if every element of A is an element of B, and every element of B is an element of A.

A less formal way of expressing this is:

„Two sets are equal if they have the same elements and the order does **not** matter.

Belongs to: This symbol '∈' is used, if a particular element is said to be belonging to a set A. If the set A = {a, b, c}, then we refer that the element a belongs to set A, as a ∈ A. And if a particular element d does not belong to the set A, then we denote it as d ∉ set A

Finite set: is the set whose end is known.

Infinite set: is the set whose end is unknown.

Empty set: {} Ø, is the set that contains nothing.

Example/ Identify the following as a set or not:

1. Months of the year.
2. Days of the week.
3. Flowers in the garden.
4. Factors of the number 6.
5. Letters of the Arabic language.
6. Short students.

Example/ Put a ⊆ or ⊄ or mark

1. {1, 2} .... {1, 5, 7}
2. {8, 9} .... {9, 8}
3. {2, 6, 7} .... {6, 7, 3}
4. {9, 7} .... {9}

Example/ Put a ∈or ∉mark

- 6 .... {6, 7}
- 8.......... {7, 4}
- 31 .... {3, 1}
- 0 .... {100, 10}

Example/ Identify the following as a Finite set or Infinite set or Ø:

Set of N

Set of Z

Set of Z between 8, 9

Set Operations

Definition// Let A and B be sets. The union of the sets A and B , denoted by A U B , is the set that contains those elements that are either in A or in B , or in both. An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that $A \cup B = \{x \mid x \in A \vee x \in B\}$. The Venn diagram shown in Figure 1 represents the union of two sets A and B. The area that represents A U B is the shaded area within either the circle representing A or the circle representing B.

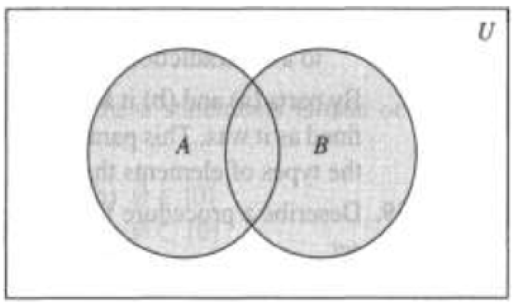


figure 1 Venn Diagram Representing the Union of A and B (A U B is shaded)

EX: The union of the sets { 1 , 3 , 5 } and { 1 , 2, 3 } is the set { 1 , 2 , 3 , 5 } ; that is, { 1 , 3 , 5 } U { 1 , 2, 3 } = { 1 , 2, 3 , 5 } .

Definition// Let A and B be sets. The **intersection** of the sets A and B , denoted by $A \cap B$, is the set containing those elements in both A and B .
An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B . This tells us that
$A \cap B = \{x \mid x \in A \wedge x \in B\}$.
The Venn diagram shown in Figure 2 represents the intersection of two sets A and B . The shaded area that is within both the circles representing the sets A and B is the area that represents the intersection of A and B .

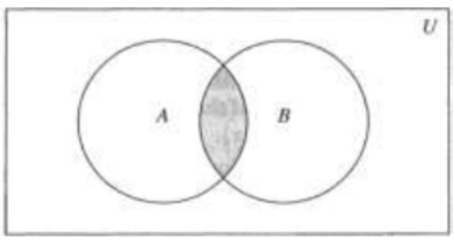


figure 2 Venn Diagram Representing the Intersection of A and B ($A \cap B$ is shaded)

EX: The intersection of the sets { 1 , 3 , 5 } and { 1 , 2 , 3 } is the set { 1 , 3 } ;
that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

Definition// Let A and B be sets. The **difference** of A and B , denoted by A - B , is the set containing those elements that are in A but not in B . The difference of A and B is also called the **complement** of B with respect to A.
An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$.
This tells us That $A - B = \{x \mid x \in A \wedge x \notin B\}$.

The **Venn diagram** shown in Figure 3 represents the difference of the sets A and B . The shaded area inside the circle that represents A and outside the circle that represents B is the area that represents A - B .

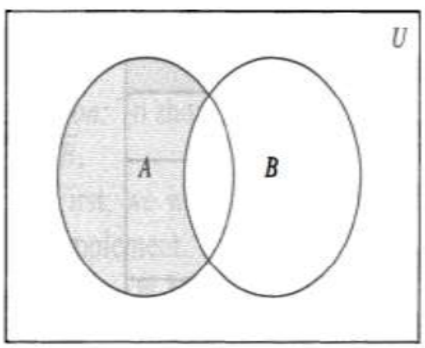
EX: The difference of { 1 , 3 , 5 } and { 1 , 2 , 3 } is the set { 5 } ; that is, { 1 , 3 , 5 } - { 1 , 2 , 3 } = { 5 } . This is different from the difference of { 1 , 2 , 3} and { 1 , 3 , 5 } , which is the set {2}.

Definition// Once the universal set U has been specified, the **complement** of a set can be defined. Let U be the universal set. The complement of the set A , denoted by Ã, is the complement of A with respect to U. In other words, the complement of the set A is U -A .

An element belongs to Ã if and only if x ∉ A . This tells us that if Ã = {x / x ∉ A } .

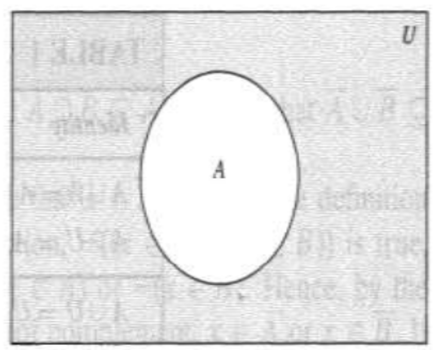
EX: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then Ã = {1, 2, 3 , 4 , 5 , 6 , 7 , 8 , 9 , 10}.

In Figure 4 the shaded area outside the circle representing A i s the area representing Ã.



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of *A* and *B*.



$\bar{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set *A*.

$A \cup \emptyset = A$ $A \cap U = A$	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
$A \cup U = U$ $A \cap \emptyset = \emptyset$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cup A = A$ $A \cap A = A$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{\overline{A}} = A$	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
$A \cap B = B \cap A$ $A \cup B = B \cup A$	$\overline{A} \cup A = U$ $\overline{A} \cap A = \emptyset$

Ex/**Prove that ($\overline{A \cup B} = \overline{A} \cap \overline{B}$)**

we must Prove:

$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

$\Rightarrow \overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

let $x \in \overline{A \cup B}$ then

$x \notin A \cup B$

$x \notin A \wedge x \notin B$

$\Rightarrow x \in \overline{A} \wedge x \in \overline{B}$

$\therefore x \in \overline{A} \cap \overline{B}$

In the same way:

$\Rightarrow \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

let $x \in \overline{A} \cap \overline{B}$ then

$x \in \overline{A} \wedge x \in \overline{B}$

$\Rightarrow x \notin A \wedge x \notin B$

$\therefore x \notin A \vee x \notin B$

$\therefore x \notin A \cup B$

$\therefore x \in \overline{A \cup B}$

Prove: $\overline{\overline{A}} = A$.

sol:

we must prove

$\overline{\overline{A}} \subseteq A$ and $A \subseteq \overline{\overline{A}}$

① $\overline{\overline{A}} \subseteq A$

let $x \in \overline{\overline{A}}$

$x \notin \overline{A}$

then $x \in A$

② $A \subseteq \overline{\overline{A}}$

let $x \in A$

$x \notin \overline{A}$

$\therefore x \in \overline{\overline{A}}$

Prove: $A - B = A \cap \bar{B}$

① $A - B$

let $x \in A - B$

$\Rightarrow x \in A \wedge x \notin B$

$\therefore x \in A \wedge x \in \bar{B}$

$\therefore x \quad x \in A \cap \bar{B}$

② $A \cap \bar{B}$

let $x \in A \wedge x \in \bar{B}$

$\Rightarrow x \in A \wedge x \notin B$

$\therefore x \in A - B$

Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an **arbitrary ordering** of the elements of the universal set. This method of representing sets makes computing combinations of sets easy. Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance a1 , a2 , . . . , an . Represent a subset A of U with the bit string of length n, where the i th bit in this string is 1 if aj belongs to A and is 0 if aj does not belong to A .

EX: Let U = { l , 2, 3 , 4, 5 , 6, 7 , 8 , 9, 10} , and the ordering of elements of U has the elements in increasing order; that is, aj = i . What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution:

The bit string that represents the set of **odd** integers i n U, namely,
{1 , 3 , 5 , 7, 9}. It is 1 0 1 0 1 0 1 0 1 0.

The bit string that represents the set of all **even** integers in U, namely,
{2, 4, 6, 8, 10}, by the string 0 1 0 1 0 1 0 1 0 1 .

The set of all integers in U that do not **exceed 5** , namely, { l , 2 , 3 , 4 , 5 } ,
is represented by the string 1 1 1 1 1 0 0 0 0 0.

NOTE// To obtain the bit string for the union and intersection of 2 sets we perform bitwise Boolean operation on bit string

Union as OR

Intersection as AND

EX// Let A1 **is** { 1 , 3, 5 , 7, 9}, A2 is { l , 2, 3, 4, 5} , Find bit string of A1 U A2, A1 ∩ A2?

Solution:

A1= { 1 , 3, 5 , 7, 9}= 1 0 1 0 1 0 1 0 1 0

A2= { l , 2 , 3 , 4, 5}= 1 1 1 1 1 0 0 0 0 0

A1 ∩ A2= 1 0 1 0 1 0 0 0 0 0

A1 U A2= 1 1 1 1 1 0 1 0 1 0

Matrices

Matrices are used throughout discrete mathematics to express relationships between elements in sets. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models.

Definition// A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an m x n matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called square. Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix.

We now introduce some terminology about matrices. Boldface uppercase letters will be used to represent matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The ith *row* of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \ldots, a_{in}]$. The jth *column* of **A** is the $n \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nj} \end{bmatrix}.$$

The (i, j)th element or entry of A is the element aij, that is, the number in the ith row and jth column of A. A convenient shorthand notation for expressing the matrix A is to write A = [aij], which indicates that A is the matrix with its (i, j)th element equal to aij.

Matrix Arithmetic

The basic operations of matrix arithmetic will now be discussed.

- **The sum of two matrix**

Definition// Let A = [aij] and B = [bij] be m x n matrices. The sum of A and B, denoted by A + B, is the m x n matrix that has aij + bij as its (i, j)th element. In other words, A + B = [aij + bij]. The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, because the sum of two matrices is defined only when both matrices have the same number of rows and the same number of columns.

We have $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$.

- **The subtract of two matrix**

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 4 & 4 \end{bmatrix}$$

- **The product of two matrix**

Definition// Let A be an m x k matrix and B be an k x n matrix. The product of A and B, denoted by AB, is the m x n matrix with its (i , J)th entry equal to the sum of the products of the corresponding elements from the ith row of A and the Jth column of B. In other words, if AB = [cii] then cij = ai1 b1j + ai2b2i + . . . + aikbki.

A product of two matrices is defined only when the number of columns in the first matrix equals the number of rows of the second matrix.

Ex/

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find AB if it is defined.

Solution: Because A is a 4 x 3 matrix and B is a 3 x 2 matrix, the product AB is defined and is a 4 x 2 matrix.

To find the elements of AB, the corresponding elements of the rows of A and the columns of B are first multiplied and then these products are added.

AB are computed, we see that

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}.$$

- **Transposes of Matrices**

Definition// Let A = [aij] be an m x n matrix. The transpose of A, denoted by N, is the n x m matrix obtained by interchanging the rows and columns of A. In other words, if At = [hij], then hij = aji for i = 1 , 2 , ... , n and j = 1 , 2, ... ,m.

EX// The transpose of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is the matrix } \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

- **Zero-One Matrices**

A matrix with entries that are either 0 or 1 is called a zero-one matrix. Algorithms using these structures are based on Boolean arithmetic with zero-one matrices. This arithmetic is based on the Boolean operations ∧ and V, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Join and Meet of the zero-one matrixes

Definition// Let A = [aij] and B = [bij] be m x n zero-one matrices. Then the **join** of A and B is the zero-one matrix with (i, j)th entry aij V bij. The join of A and B is denoted by A V B. The **meet** of A and B is the zero-one matrix with (i, j)th entry aij ∧ bij. The meet of A and B is denoted by A ∧B.

EX// Find the **join** and **meet** of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We now define the **Boolean product** of two matrices.

- **Boolean product of the zero-one matrixes**

Definition// Let A = [aij] be an m x k zero-one matrix and B = [bij] be an k x n zero one matrix. Then the Boolean product of A and B, denoted by AOB, is the mxn matrix with (i, j)th entry Cij where Cij = (ail Λ b1j)V (ai2 Λ b2j) V ... V (aik Λ bkj). Note that the Boolean product of A and B is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation V and with multiplication replaced with the operation Λ. We give an example of the Boolean products of matrices.

EX// Find the Boolean product of A and B, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Propositional and Logical Operations

Propositional Logic

A proposition is a declarative sentence that is either true or false, but not both.

EX// Consider the following sentences.

1 . What time is it?

2 . Read this carefully.

3 . x + 1 = 2 .

4 . x + y = Z.

5. Baghdad is the capital of Iraq.

6. 10+20=40.

Solution:

Sentences 1 and 2 are not propositions because they are not declarative sentences.

Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables. Sentence 5 is true proposition but Sentence 6 is false proposition.

- Many mathematical statements are constructed by combining one or more propositions. New propositions, called **compound propositions**, are formed from existing propositions using logical operators.

TABLE 1 The Truth Table for the Negation of a Proposition.

P	¬p
T	F

EX//
"Today is Friday." → P
"Today is NOT Friday." → ¬P

Definition // Let p and q be propositions. The **conjunction** of p and q , denoted by $p \wedge q$, is the proposition "p and q ". The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Table 2 displays the truth table for $p \wedge q$.

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition// Let p and q be propositions. The **disjunction** of p and q, denoted by $p \vee q$, is the proposition "p or q ". The disjunction $(p \vee q)$ is **false** when both p and q are false and is **true** otherwise.

Table 3 displays the truth table for $p \vee q$.

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition// Let p and q be propositions. The exclusive or of p and q, denoted by $p \oplus q$, is the proposition that is **true** when exactly one of p and q is true and is **false** otherwise. The truth table for the exclusive or of two propositions is displayed in Table 4.

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

Definition// Let p and q be propositions. The conditional statement p →q is the proposition "if p, then q ". The conditional statement p→ q is false when p is true and q is false, and true otherwise. The truth table for the conditional statement p→ q is shown in Table 5.

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition// Let p and q be propositions. The biconditional statement p ↔ q is the proposition "p if and only if q ." The biconditional statement p ↔ q is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called **bi-implications**.

- The last way of expressing the bi-conditional statement p → q uses the abbreviation "iff" for "if and only if". Note that p ↔ q has exactly the same truth value as (p → q) ∧ (q → p).

The truth table for p ↔ q is shown in Table 6.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EX//Construct the truth table of the compound proposition
$(p \vee \neg q) \rightarrow (p \wedge q)$.
The resulting truth table is shown in Table 7.

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Propositional Equivalences

Definition// A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a **tautology**. A compound proposition that is always false is called a **contradiction.** A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

- **Tautology** $(p \vee \neg p)$, **Contradiction** $(p \wedge \neg p)$

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Definition // The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

EX// $p \to q \equiv \neg p \vee q$

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

EX// is $(p \wedge q) \wedge \neg(p \vee q)$ tautology or contradiction

$$
\begin{aligned}
(p \wedge q) \wedge \neg(p \vee q) &= (p \wedge q) \wedge (\neg p \wedge \neg q) \\
&= p \wedge q \wedge \neg p \wedge \neg q \\
&= (p \wedge \neg p) \wedge (q \wedge \neg q) \\
&= F \wedge F \\
&= F \to \text{contradiction}
\end{aligned}
$$

Quantifiers

The universal quantification of P (x) is the statement "P(x) for all values

of x in the domain." The notation ∀x P (x) denotes the universal quantification of P(x). Here ∀ is called the universal quantifier. We read ∀x P(x) as "for all x P(x)" or "for every x P(x)." An element for which P(x) is false is called a counterexample of x P (x). The meaning of the universal quantifier is summarized in the first row of Table 1.

Definition// The existential quantification of P(x) is the proposition

"There exists an element x in the domain such that P (x)." We use the notation ∃x P(x) for the existential quantification of P(x). Here ∃ is called the existential quantifier.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Ex\ $\forall x \epsilon N; x \geq 0$ ➔ True

$\forall x \epsilon R; x \geq 0$ ➔ False

∃x $x \epsilon R; a^2 - 1 = 0$ ➔ True

∃x $x \epsilon R; a^2 + 1 = 0$ ➔ False

EX// Some student in the class has visited cairo, and every one in the class has visited either Baghdad or cairo

Solution:

P(x): visited cairo, Q(x): visited baghdad , $\exists x\ p(x) \wedge \forall x\ (p(x) \vee q(x))$

EX// The sum of two positive integers is positive ?

Solution:

$\forall X\ \forall y\ (x>0 \wedge y>0) \rightarrow x+y>0$

EX// Let $p(x) \equiv x>3$ what are the truth values of p(4) and p(2)

Solution:

P(4) true, P(2) false

EX// Let $q(x,y) \equiv x=y+3$, what are the truth value of the propositions q(1,2), q(3,0).

Solution:

q(1,2) false, q(3,0) true

Negating Quantified Expressions

The negation for $\exists x\ p(x)$ is $\forall x \neg p(x)$

The negation for $\forall x\ p(x)$ is $\exists x \neg p(x)$

EX// There is student in your class who has taken a course in calculus: $\exists x\ p(x)$

Every student in your class, has not taken a course in calculus: $\forall\ x \neg p(x)$

$\sim (\forall \; x \in A \; ; \; P(x))$

$\exists \; x \in A \; ; \sim P(x)$

$\sim (\exists \; x \in A \; ; \; P(x))$

$\forall \; x \in A \; ; \; \sim P(x)$

Ex\ $\forall \; x \in R \; ; \; a^2 - 1 \geq 0$

sol \ $\exists \; x \in R \; ; \; a^2 - 1 < 0$

Ex \ $\exists \; x \in R ; \forall x \in N ; \; x = y^2$

sol :- $\forall x \in R ; \exists x \in N ; \; x \neq y^2$