Power Mechanics Technical Engineering Department

Second Stage

Fluid Mechanics - Dynamic

Chapter One: Fluid Kinematics

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Fluid Kinematics

1. Introduction

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. The fluid in motion is described by two methods, they are

i. ***Lagrangian Method***
ii. ***Eulerian Method***

In the Lagrangian method, a **single fluid particle** is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, pressure, density, etc., are described at **a point** in flow field. The Eulerian method is commonly used in fluid mechanics.

2. Types of Fluid Flow

The fluid flow is classified as:

i. *Steady and Unsteady flows;*
ii. *Uniform and non-uniform flows;*
iii. *Laminar and Turbulent flows;*
iv. *Compressible and Incompressible flows;*
v. *Rotational and Irrotational flows;*
vi. *Viscous and Inviscid flows; and*
vii. *One-, Two-, and Three-Dimensional flows.*

2.1 Steady and Unsteady Flows

Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time.

$$\left(\frac{\partial V}{\partial t}\right)_{x_0,\,y_0,\,z_0} = 0,\ \left(\frac{\partial p}{\partial t}\right)_{x_0,\,y_0,\,z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0,\,y_0,\,z_0} = 0$$

Unsteady flow is that type of flow, in which the velocity, pressure of density at a point changes with respect to time.

$$\left(\frac{\partial V}{\partial t}\right)_{x_0,\,y_0,\,z_0} \neq 0,\ \left(\frac{\partial p}{\partial t}\right)_{x_0,\,y_0,\,z_0} \neq 0 \text{ etc.}$$

2.2 Uniform and Non-uniform Flows

Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (length of direction of the flow).

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} = 0$$

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space.

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} \neq 0.$$

2.3 Laminar and Turbulent Flows

Laminar flow is defined as that type of flow in which the fluid particles move along well defined paths or stream line and all the stream-lines are straight and parallel.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in *zig-zag* way, the eddies formation takes place which are responsible for high energy loss.

For a pipe flow, the type of flow is determined by a non-dimensional number called the **Reynold number**.

$$Re = \frac{\rho\, V\, D}{\mu} = \frac{V\, D}{\nu}$$

Where ρ, μ, and ν, is the density, absolute viscosity, and kinematic viscosity of fluid respectively. *V* is the mean velocity of the flow, and *D* is diameter of pipe.

If Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

2.4 Compressible and Incompressible Flows

Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid.

$$\rho \neq Constant$$

Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible.

$$\rho = Constant$$

2.5 Rotational and Irrotational Flows

Rotational flow is that flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow called *Irrotational flow*.

2.6 Viscous and Inviscid Flows

Flows in which the effects of viscosity are significant are called *viscous flows*. The effects of viscosity are very small in some flows, and neglecting those effects greatly simplifies the analysis without much loss in accuracy. Such idealized flows of zero-viscosity fluids are called frictionless or *inviscid flows*.

2.7 One-, Two-, and Three-Dimensional Flows

A flow field is best characterized by the velocity distribution, and thus a flow is said to be one-, two-, or three-dimensional if the flow velocity varies in one, two, or three primary dimensions. For one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$

Where u, v, and w are velocity components in x, y, and z directions respectively. For two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$

For three-dimensional flow

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

3. Continuity Equation

The equation based on the principle of conservation of mass is called continuity equation. The **conservation of mass principle** for a system can be expressed as: *The net mass transfer to or from a system during a time interval Δt is equal to the net change (increase or decrease) in the total mass within the system during Δt*. That is,

$$m_{in} - m_{out} = \Delta m_{sys} \qquad (kg)$$

Where m_{in} and m_{out} are the mass flow into and out of the system. During a **steady flow** process, the total amount of mass contained within a system does not change with time ($m_{sys} =$ constant). Then the conservation of mass principle requires that the total amount of mass entering a control volume equal the total amount of mass leaving it.

$$\sum_{in} m = \sum_{out} m$$

The quantity of a fluid per second through a section of a pipe or channel is defined as ***rate of flow*** or ***discharge***. For an incompressible fluid (liquid) the rate of flow is expressed as the volume of fluid flowing across the section per second (**volume flow rate**). For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section.

i. For liquids the units of **Q** are m^3/s or liters/s.
ii. For gases the units of **Q** is *N/s*.

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section, then discharge

$$\boldsymbol{Q = A \times V}$$

The amount of mass flowing through a cross section per unit time is called the ***mass flow rate*** and is denoted by $\dot{m}$. The dot over a symbol is used to indicate *time rate of change*.

$$\dot{m} = \rho \times A \times V$$

where ρ is the density of the fluid flow. The mass and volume flow rates are related by

$$\dot{m} = \rho \times Q = \frac{Q}{v}$$

where v is the specific volume ($v = 1/\rho$).

For a fluid flowing through the pipe at all the cross- section, the quantity of fluid per second is constant ($\dot{m} = constant$). Consider two cross-sections of a pipe as shown in Fig.1.

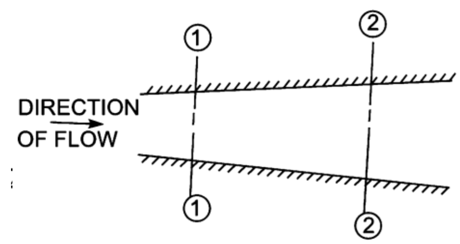


Fig.1 fluid flowing through a pipe.

Let V_1 = Average velocity at cross-section 1-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section 1-1

and V_2, ρ_2, A_2 are corresponding values at section, 2-2.

Then mass flow rate at section 1-1 = $\rho_1 A_1 V_1$

Mass flow rate at section 2-2 = $\rho_2 A_2 V_2$

According to law of conservation of mass

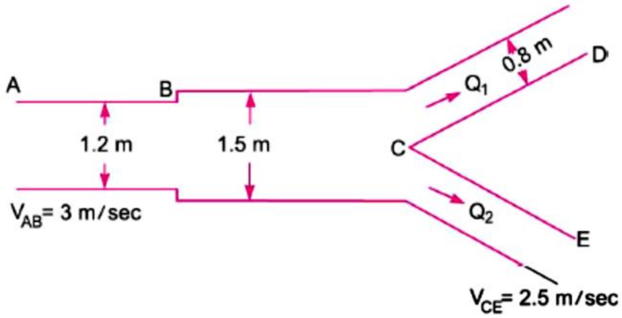
$$Mass\ of\ flow\ at\ section\ 1-1 = Mass\ of\ flow\ at\ section\ 2-2$$

$$\boldsymbol{\rho_1 A_1 V_1 = \rho_2 A_2 V_2}$$

This equation is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation reduces to

$$A_1 V_1 = A_2 V_2$$

<u>Example (1):</u> *Water flows through a pipe AB 1.2 m diameter at 3 m/s and then passes through a pipe BC 1.5 m diameter. At C, the pipe branches. Branch CD is 0.8 m in diameter and carries one- third of the flow in AB. The flow velocity in branch CE is 2.5* *m/s. Find the volume rate of flow in AB, the velocity in BC, the velocity in CD and the diameter of CE.*



Solution: Given:

Diameter of pipe *AB*, $D_{AB} = 1.2$ m

Velocity of flow through *AB*, $V_{AB} = 3.0$ m/s

Dia. of pipe *BC*, $D_{BC} = 1.5$ m

Dia. of branched pipe *CD*, $D_{CD} = 0.8$ m

Velocity of flow in pipe *CE*, $V_{CE} = 2.5$ m/s

Let the flow rate in pipe $AB = Q$ m³/s

Velocity of flow in pipe $BC = V_{BC}$ m/s

Velocity of flow in pipe $CD = V_{CD}$ m/s

Diameter of pipe $CE = D_{CE}$

Then flow rate through $CD = Q/3$

and flow rate through $CE = Q - Q/3 = \frac{2Q}{3}$

(*i*) Now volume flow rate through $AB = Q = V_{AB} \times$ Area of AB

$$= 3.0 \times \frac{\pi}{4}(D_{AB})^2 = 3.0 \times \frac{\pi}{4}(1.2)^2 = \mathbf{3.393\ m^3/s.\ Ans.}$$

(*ii*) Applying continuity equation to pipe *AB* and pipe *BC*,

$$V_{AB} \times \text{Area of pipe } AB = V_{BC} \times \text{Area of pipe } BC$$

$$3.0 \times \frac{\pi}{4}(D_{AB})^2 = V_{BC} \times \frac{\pi}{4}(D_{BC})^2$$

$$3.0 \times (1.2)^2 = V_{BC} \times (1.5)^2$$

$$V_{BC} = \frac{3 \times 1.2^2}{1.5^2} = \mathbf{1.92\ m/s.\ Ans.}$$

(*iii*) The flow rate through pipe

$$CD = Q_1 = \frac{Q}{3} = \frac{3.393}{3} = 1.131 \text{ m}^3/\text{s}$$

$$\therefore \quad Q_1 = V_{CD} \times \text{Area of pipe } CD \times \frac{\pi}{4}(D_{CD})^2$$

$$1.131 = V_{CD} \times \frac{\pi}{4} \times 0.8^2 = 0.5026\, V_{CD}$$

$$\therefore \quad V_{CD} = \frac{1.131}{0.5026} = \mathbf{2.25 \text{ m/s. Ans.}}$$

(*iv*) Flow rate through *CE*,

$$Q_2 = Q - Q_1 = 3.393 - 1.131 = 2.26$$

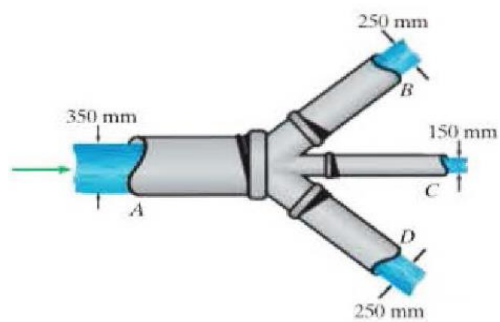
$$\therefore \quad Q_2 = V_{CE} \times \text{Area of pipe } CE = V_{CE}\frac{\pi}{4}$$

$$2.263 = 2.5 \times \frac{\pi}{4} \times (D_{CE})^2$$

$$D_{CE} = \sqrt{\frac{2.263 \times 4}{2.5 \times \pi}} = \sqrt{1.152} = 1.0735$$

$\therefore$ Diameter of pipe CE = **1.0735 m. Ans.**

Q/ Water flows through the pipe at A at 300 kg/s, and then out the double wye with an average velocity of 3 m/s through B and an average velocity of 2 m/s through C. Determine the average velocity at which it flows through D.



4. Equations of Motion

Dynamics of fluid flow is the study of fluid motion with the forces causing flow. The dynamic behavior of the fluid flow is analyzed by the Newton's second law of motion, which relates the acceleration with the forces.

According to Newton's second law of motion, *the net force F_x acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a_x in the x-direction*. Thus mathematically.

$$F_x = m\, a_x$$

In the fluid flow, the following forces are present:

i. F_g, gravity force.

ii. F_p, the pressure force.

iii. F_v, force due to viscosity.

iv. F_t, force due to turbulence.

v. F_c, force due to compressibility.

Thus, the net force

$$F_x = \left(F_g\right)_x + \left(F_p\right)_x + (F_v)_x + (F_t)_x + (F_c)_x$$

a) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = \left(F_g\right)_x + \left(F_p\right)_x + (F_v)_x + (F_t)_x$$

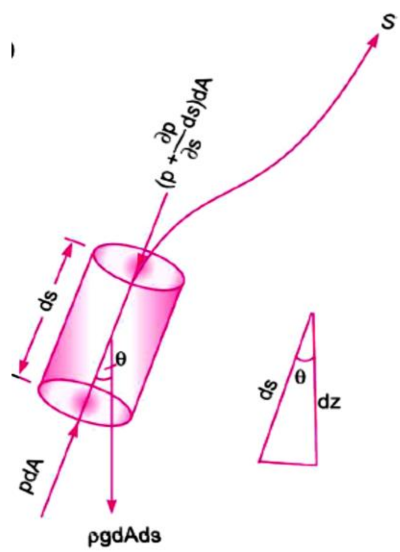
and equation of motions are called **Reynold's equations of motion**,

b) For flow, where F_t is negligible, the resulting equations of motion are known as **Navier Stokes Equation**.

c) If the flow is assumed to be ideal, viscous force *(F_v)* is zero and equation of motions are known as **Euler's equation of motion**.

5. Euler's Equation of Motion

This is equation of motion in which the forces due to **gravity and pressure** are taken into consideration. This is derived by considering the motion of a fluid element along a stream- line as shown in *Fig.* Consider a cylindrical element of cross-section *dA* and length *ds*. The forces acting on the cylindrical element are:



1. *Pressure force* ***pdA*** *in the direction of flow.*
2. *Pressure force* $\left(\boldsymbol{p} + \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{s}} \boldsymbol{ds}\right) \boldsymbol{dA}$ *opposite to the direction of flow.*
3. *Weight of element* ***ρgdAds***.

The resultant force on the fluid element in the direction of *s* must be equal to the mass of fluid element multiplied by acceleration in the direction *s*. and after arrangements

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + g\,\frac{dz}{ds} + v\frac{\partial v}{\partial s} = 0$$

$$\boxed{\frac{\boldsymbol{dp}}{\boldsymbol{\rho}} + \boldsymbol{g}\,\boldsymbol{dz} + \boldsymbol{vdv} = \mathbf{0}}$$

This equation is known as ***Euler's equation of motion.***

6. Bernoulli's Equation

Bernoulli's equation is *an approximate relation between pressure, velocity, and elevation*.

The assumptions made in the derivation of Bernoulli's equation:

i. *The fluid is ideal, viscosity is zero*
ii. *The flow is steady*
iii. *The flow is incompressible*
iv. *The flow is irrotational.*

Bernoulli equation is obtained by integrating the *Euler's equation of motion as:*

$$\int \frac{dp}{\rho} + \int g\,dz + \int vdv = constant$$

If flow is incompressible, ρ is constant and

$$\frac{p}{\rho} + gz + \frac{v^2}{2} = constant \quad or \quad \frac{p}{\rho g} + z + \frac{v^2}{2g} = constant$$

$$\boxed{\frac{\boldsymbol{p}}{\boldsymbol{\rho g}} + \frac{\boldsymbol{v^2}}{\boldsymbol{2g}} + \boldsymbol{z} = \boldsymbol{constant}}$$

This equation is ***Bernoulli's equation*** in which

$\frac{p}{\rho g} = pressure\ energy\ per\ unit\ weight\ of\ fluid\ or\ \boldsymbol{pressure\ head};$

$(it\ represents\ the\ height\ of\ a\ fluid\ column\ that\ produces\ the\ static\ pressure\ P)$

$\frac{v^2}{2g} = kinetic\ energy\ per\ unit\ weight\ or\ \boldsymbol{kinetic\ head}; (it\ represents\ the$

$elevation\ needed\ for\ a\ fluid\ to\ reach\ the\ velocity\ V\ during\ frictionless\ free\ fall)$

$z = potential\ energy\ per\ unit\ weight\ or\ \boldsymbol{evelation\ head.}$

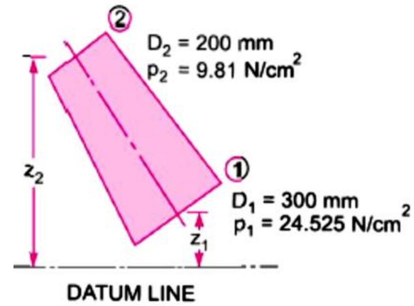
The summation of these three heads represent **Total Head** (*H*).

The Bernoulli equation can also be written between any two points on the same streamline as

$$\frac{(p_2 - p_1)}{\rho} + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = constant$$

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = constant$$

Example (2): *Water is flowing through a pipe having diameter 300 mm and 200 mm at the bottom and upper end respectively. The intensity of pressure at the bottom end is 24.525 N/cm^2 and the pressure at the upper end is 9.81 N/cm^2. Determine the difference in datum head if the rate of flow through pipe is 40 lit/s.*



Solution:

Section 1, $D_1 = 300 \text{ mm} = 0.3 \text{ m}$

$p_1 = 24.525 \text{ N/cm}^2 = 24.525 \times 10^4 \text{ N/m}^2$

Section 2, $D_2 = 200 \text{ mm} = 0.2 \text{ m}$

$p_2 = 9.81 \text{ N/cm}^2 = 9.81 \times 10^4 \text{ N/m}^2$

Rate of flow $= 40$ lit/s

or $$Q = \frac{40}{1000} = 0.04 \text{ m}^3/\text{s}$$

Now $$A_1V_1 = A_2V_2 = \text{rate of flow} = 0.04$$

$$\therefore \quad V_1 = \frac{.04}{A_1} = \frac{.04}{\frac{\pi}{4}D_1^2} = \frac{0.04}{\frac{\pi}{4}(0.3)^2} = 0.5658 \text{ m/s}$$

$$\simeq 0.566 \text{ m/s}$$

$$V_2 = \frac{.04}{A_2} = \frac{.04}{\frac{\pi}{4}(D_2)^2} = \frac{0.04}{\frac{\pi}{4}(0.2)^2} = 1.274 \text{ m/s}$$

Applying Bernoulli's equation at sections (1) and (2), we get

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

or $$\frac{24.525\times10^4}{1000\times9.81}+\frac{.566\times.566}{2\times9.81}+z_1=\frac{9.81\times10^4}{1000\times9.81}+\frac{(1.274)^2}{2\times9.81}+z_2$$

or $$25+.32+z_1=10+1.623+z_2$$

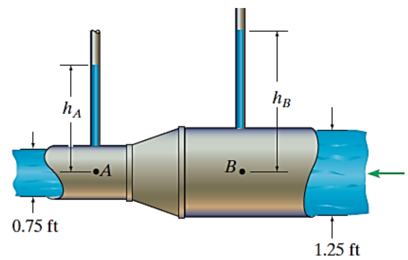
or $$25.32+z_1=11.623+z_2$$

$$\therefore\quad z_2-z_1=25.32-11.623=13.697=13.70\text{ m}$$

$\therefore$ Difference in datum head $= z_2 - z_1 = \mathbf{13.70\ m.\ Ans.}$

Example (3): *Water flows up through the vertical pipe such that when it is at A, it is subjected to a pressure of 150 kPa and has a velocity of 3 m/s. Determine the pressure and its velocity at B. Set d = 75 mm.*

Solution:



Continuity Equation.

$$\frac{d}{dt}\int_{cv} e\, d\mathcal{V} + \int_{cs} e\mathbf{V}\cdot d\mathbf{A} = 0$$

$$0 - V_A A_A + V_B A_B = 0$$

$$-(3\text{ m/s})\left[\pi(0.05\text{ m})^2\right] + V_B\left[\pi(0.0375\text{ m})^2\right] = 0$$

$$V_B = 5.333\text{ m/s} = 5.33\text{ m/s} \qquad \textbf{Ans.}$$

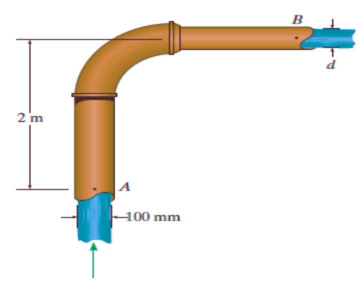
Bernoulli Equation. If we set the datum at point A, $Z_A = 0$ and $z_B = 2$ m.

$$\frac{p_A}{\rho}+\frac{V_A^2}{2}+gz_A=\frac{p_B}{\rho}+\frac{V_B^2}{2}+gz_B$$

$$\left[\frac{150(10^3)\text{ N/m}^2}{1000\text{ kg/m}^3}\right]+\frac{(3\text{ m/s})^2}{2}+0=\frac{p_B}{(1000\text{ kg/m}^3)}+\frac{(5.333\text{ m/s})^2}{2}+(9.81\text{ m/s}^2)($$

$$p_B = 120.66(10^3)\text{ Pa} = 121\text{ kPa} \qquad \textbf{Ans.}$$

Example (4): *The volumetric flow of water through the transition is 3 ft^3/s. Determine the height it rises in the piezometer at A if hB = 2 ft.*



SOLUTION

The control volume considered contains the water within the transition from A to B which is fixed. Using the discharge,

$$Q = V_A A_A; \qquad 3\text{ ft}^3/\text{s} = V_A\left[\pi\left(\frac{0.75\text{ ft}}{2}\right)^2\right] \qquad V_A = 6.7906\text{ ft/s}$$

$$Q = V_B A_B; \qquad 3\text{ ft}^3/\text{s} = V_B\left[\pi\left(\frac{1.25\text{ ft}}{2}\right)^2\right] \qquad V_B = 2.4446\text{ ft/s}$$

Since water can be considered as an ideal fluid (incompressible and inviscid) and the flow is steady, Bernoulli's equation is applicable. Writing this equation between points B and A,

$$\frac{p_B}{\rho_w}+\frac{V_B^2}{2}+gz_B=\frac{p_A}{\rho_w}+\frac{V_A^2}{2}+gz_A$$

The static pressures at A and B are

$$p_A = \rho_w g h_A = \rho_w (32.2\ \text{ft/s}^2) h_A = (32.2\ \rho_w h_A)\ \text{lb/ft}^2$$

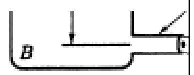
$$p_B = \rho_w g h_B = \rho_w (32.2\ \text{ft/s}^2)\ (2\ \text{ft}) = (64.4\ \rho_w)\ \text{lb/ft}^2$$

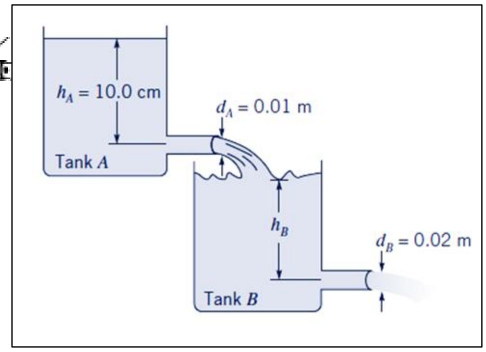
If we set the datum to coincide with the horizontal streamline connecting A and B, then $z_A = z_B = 0$. Therefore,

$$\frac{64.4\ \rho_w}{\rho_w} + \frac{(2.4446\ \text{ft/s})^2}{2} + 0 = \frac{32.2\ \rho_w h_A}{\rho_w} + \frac{(6.7906\ \text{ft/s})^2}{2} + 0$$

$$h_A = 1.377\ \text{ft} = 1.38\ \text{ft} \qquad \textbf{Ans.}$$

Example (5): *Find the water height hB in tank B shown in Fig. for steady-state conditions.*





For steady flow, $Q_2 = Q_4$ where

$$Q_4 = A_4 V_4 \quad \text{with} \quad \frac{p_3}{\gamma} + \frac{V_3^2}{2g} + z_3 = \frac{p_4}{\gamma} + \frac{V_4^2}{2g} + z_4$$

where $p_3 = p_4 = 0$ and $V_3 = 0$

Thus,

$$V_4 = \sqrt{2g\,(z_3 - z_4)} = \sqrt{2(9.81\ \tfrac{m}{s^2})(2m)} = 6.26\ \frac{m}{s}$$

or

$$Q_4 = \frac{\pi}{4}(0.05m)^2 (6.26\ \tfrac{m}{s}) = 0.0123\ \frac{m^3}{s}$$

Also,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \quad \text{where } p_1 = p_2 = 0 \text{ and } V_1 = 0$$

so that

$$V_2 = \sqrt{2g h_A}$$

Thus,

$$A_2 V_2 = Q_4 \quad \text{or} \quad \frac{\pi}{4}(0.03m)^2 \sqrt{2(9.81\ \tfrac{m}{s^2}) h_A} = 0.0123\ \frac{m^3}{s}$$

or

$$h_A = \underline{\underline{15.4\ m}}$$

References

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6. The Velocity Field

Velocity is a vector function of position and time and thus has three components u, v, and w, along x, y, and z directions respectively, in rectangular coordinate system.

$$\mathbf{V} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$$

Where motion of fluid is specified by velocity components expressed as function of space and time; $\mathbf{V} = \mathbf{V}(x, y, z, t)$, $u=u(x, y, z, t)$, $v=v(x, y, z, t)$, $w=w(x, y, z, t)$

$$\mathbf{V}(x, y, z, t) = u(x, y, z, t)\,\boldsymbol{i} + v(x, y, z, t)\,\boldsymbol{j} + w(x, y, z, t)\,\boldsymbol{k}$$

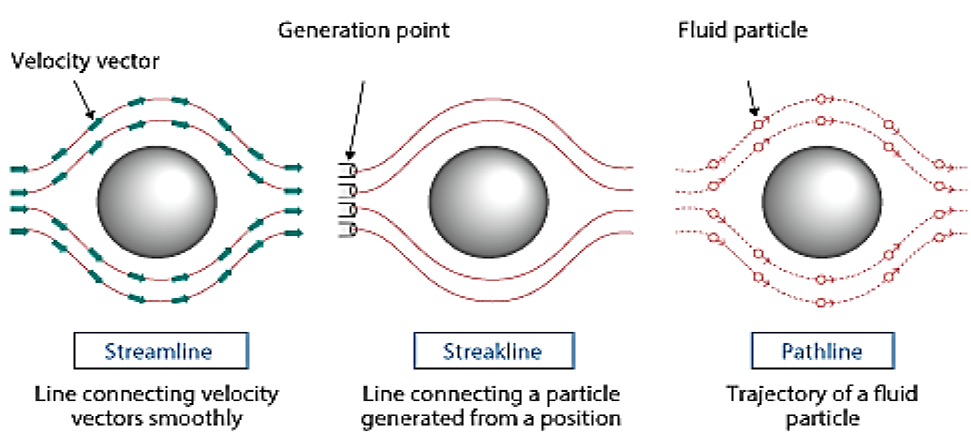
Streamline: the line that is everywhere tangent to the velocity field. If the flow is steady, nothing at a fixed point changes in time. In an unsteady flow the streamlines due change in time.

For 2D flows, integrate the equations defining lines tangent to the velocity field:

$$\frac{dy}{dx} = \frac{v}{u}$$

Streaklines: a laboratory tool used to obtain instantaneous photographs of marked particles that all passed through a given flow field at some earlier time. Neutrally buoyant smoke, or dye is continuously injected into the flow at a given location to create the picture.

Pathlines: line traced by a given particle as it flows from one point to another. This method is a Lagrangian technique in which a fluid particle is marked and then the flow field is produced by taking a time exposure photograph of its movement.



7. Acceleration Field

The acceleration of the fluid particle is the time derivative of the particle's velocity

$$\boldsymbol{a}\,(x,y,z,t)=\frac{d\boldsymbol{V}}{dt}=\frac{\partial \boldsymbol{V}}{\partial t}+\left(u\,\frac{\partial \boldsymbol{V}}{\partial x}+v\frac{\partial \boldsymbol{V}}{\partial y}+w\frac{\partial \boldsymbol{V}}{\partial z}\right)=\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}.\boldsymbol{\nabla})\boldsymbol{V}$$

The term $\partial \boldsymbol{V}/\partial t$ is called the ***local acceleration,*** which vanishes if the flow is steady. The three terms in parentheses are called the ***convective acceleration.***

Convective acceleration represents the fact the flow property associated with a fluid particle may vary due to the motion of the particle from one point in space to another.

Note our use of the compact dot product involving **V** and the gradient operator **∇**:

$$u\,\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}=\boldsymbol{V}.\boldsymbol{\nabla}\qquad \text{where}\qquad \boldsymbol{\nabla}=\mathbf{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z}$$

In Cartesian coordinates then, the components of the acceleration vector are:

$$a_x=\frac{Du}{Dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t}+\nabla.(u\boldsymbol{V})$$

$$a_y=\frac{Dv}{Dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{\partial v}{\partial t}+\nabla.(v\boldsymbol{V})$$

$$a_z=\frac{Dw}{Dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{\partial w}{\partial t}+\nabla.(w\boldsymbol{V})$$

The total acceleration,

$$a=a_x\boldsymbol{i}+a_y\,\boldsymbol{j}\,+a_z\boldsymbol{k}$$

Example (1): An idealized velocity field is given by the formula

$$\mathbf{V}=4tx\mathbf{i}-2t^2y\mathbf{j}+4xz\mathbf{k}$$

Is this flow field steady or unsteady? Is it two- or three-dimensional? At the point (x, y, z) = (–1, +1, 0), compute the acceleration vector.

Solution: The flow is unsteady because time t appears in the components. The flow is three-dimensional because all three velocity components are nonzero.

Evaluate, by laborious differentiation, the acceleration vector at (x, y, z) = (−1, +1, 0).

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=4x+4tx(4t)-2t^2y(0)+4xz(0)=4x+16t^2x$$

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-4ty+4tx(0)-2t^2y(-2t^2)+4xz(0)=-4ty+4t^4y$$

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0+4tx(4z)-2t^2y(0)+4xz(4x)=16txz+16x^2z$$

$$\text{or:}\quad \frac{d\mathbf{V}}{dt}=(4x+16t^2x)\mathbf{i}+(-4ty+4t^4y)\mathbf{j}+(16txz+16x^2z)\mathbf{k}$$

at $(x, y, z)=(-1, +1, 0)$, we obtain $\quad \dfrac{d\mathbf{V}}{dt}=-4(1+4t^2)\mathbf{i}-4t(1-t^3)\mathbf{j}+0\mathbf{k}$

Example (2): *In a fluid, the velocity field is given by*

$$V = (3x+2y)\,i + (2z+3x^2)\,j + (2t-3z)\,k$$

Determine:

(i) The velocity components u, v, w at any point in the flow field;

(ii) The speed at point (1, 1, 1);

(iii) The speed at time t = 2s at point (0, 0, 2).

Also classify the velocity field as steady, or unsteady, uniform or non-uniform and one, two or three dimensional.

Solution. *Given:* Velocity field, $V = (3x+2y)\,i + (2z+3x^2)\,j + (2t-3z)\,k$

(i) Velocity components:

The velocity components are:

$$\mathbf{u = 3x+2y,\ v=(2z+3x^2),\ w=(2t-3z)}\quad \textbf{(Ans.)}$$

(ii) Speed at point (1, 1, 1), $V_{(1,1,1)}$:

Substituting $x=1$, $y=1$, $z=1$ in the expressions for u, v and w, we have:

$$u = (3+2)=5,\ v=(2+3)=5,\ w=(2t-3)$$

$$\therefore\quad V^2 = u^2+v^2+w^2$$

$$= 5^2+5^2+(2t-3)^2$$

$$= 25+25+4t^2-12t+9$$

$$= 4t^2-12t+59$$

$$\therefore\quad V_{(1,1,1)}=\sqrt{4t^2-12t+59}\quad \textbf{(Ans.)}$$

(*iii*) Speed at $t = 2s$ at point (0, 0, 2):

Substituting $t = 2$, $x = 0$, $y = 0$, $z = 2$ in the expressions for u, v and w, we get:

$$u = 0,\ v = (2 \times 2 + 0) = 4,\ w = (2 \times 2 - 3 \times 2) = -2$$

$\therefore$ $$V^2 = u^2 + v^2 + w^2 = 0 + 4^2 + (-2)^2 = 20$$

or, $$\mathbf{V_{(0,0.2)} = \sqrt{20} = 4.472 \text{ units}} \textbf{ (Ans.)}$$

Velocity field, type:

(*i*) Since V at given (x, y, z) *depends on t* it is **unsteady flow, (Ans.)**

(*ii*) Since at given t velocity changes in the X direction it is **non-uniform flow. (Ans.)**

(*iii*) Since V depends on x, y, z it is three **dimensional flow. (Ans.)**

Example (3): *Find the velocity and acceleration at a point (1, 2, 3) after 1 sec. for a three-dimensional flow given by $u = yz + t$, $v = xz - t$, $w = xy$ m/s.*

Solution. *Given*: Three-dimensional flow field is given as:

$$u = yz + t,\ v = xz - t,\ w = xy \text{ m/s}$$

Velocity at a point 1, 2, 3 $V_{(1, 2, 3)}$:

Velocity at a point (1, 2, 3) after 1s is calculated as follows:

$$u = yz + t = 2 \times 3 + 1 = 7 \text{ m/s},\ v = xz - t = 1 \times 3 - 1 = 2 \text{ m/s and}$$

$$w = xy = 1 \times 2 = 2 \text{ m/s.}$$

$\therefore$ $$V_{(1,2,3)} = 7i + 2j + 2k$$

$$= \sqrt{7^2 + 2^2 + 2^2} = 7.55 \text{ m/s}$$

Hence, $$V_{(1, 2, 3)} = \mathbf{7.55 \text{ m/s}} \textbf{ (Ans.)}$$

Acceleration, $a_{(1,2,3)}$:

Now, $$V = (yz + t)i + (xz - t)j + xy\,k \text{ m/s}$$

Acceleration, $$a = \frac{dV}{dt} = \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right) + \frac{\partial V}{\partial t}$$

$$a = (yz + t)(zi + yk) + (xz - t)(zi + xk) + xy(yi + xj) + (1i - 1j)$$

$\therefore$ $$a_{(1, 2, 3)} = 7(3j + 2k) + 2(3i + 1k) + 2(2i + 1j) + (1i - 1j)$$

$$= (21j + 14k) + (6i + 2k) + (4i + 2j) + (1i - 1j)$$

$$= (10i + 23j + 16k) + (1i - 1j)$$

The *convective acceleration components are*: (10, 23, 16) m/s^2

The *local acceleration components are*: (1, –1) m/s^2 along x and y directions.

The total acceleration of fluid particles at the points (1, 2, 3) is given by:

$$a_{(1, 2, 3)} = \sqrt{(10 + 1)^2 + [23 + (-1)]^2 + 16^2}$$

$$= \sqrt{11^2 + 22^2 + 16^2} = 29.34 \text{ m/s}^2$$

Hence, $$a_{(1, 2, 3)} = \mathbf{29.34 \text{ m/s}^2} \textbf{ (Ans.)}$$

Example (4): *Obtain the equation to the streamlines for the velocity field given as:*

$$V = 2x^3 i - 6x^2 yj$$

Solution. *Given*: Velocity field, $V = 2x^3 i - 6x^2 yj$

Here, $u = 2x^3,\ v = 6x^2 y$

The streamlines in two dimensions are defined by:

$$\frac{dx}{u} = \frac{dy}{v}$$

or, $$\frac{dy}{dx} = \frac{v}{u} = \frac{-6x^2 y}{2x^3} = \frac{-3y}{x}$$

Separating the variables, we have:

$$\frac{dy}{y} = \frac{-3dx}{x}$$

Integrating, we get:

$$ln(y) = -3ln(x) + C_1$$

or, $$ln(y) + 3\,ln(x) = C_1$$

or, $$\mathbf{yx^3 = C}\ \textbf{(Ans.)}$$

Example (5): *For the following flows find the equation of the streamline passing through* (2,2):

(*i*) $V = 3\,xi - 3yj$

(*ii*) $V = -y^2 i - 6xj$

Solution. Equation of the streamline passing through (2, 2):

(*i*) $$V = 3xi - 3yj$$

$$u = 3x \text{ and } v = -3y$$

The equation of a streamline in two-dimensional flow is given as:

$$\frac{dx}{u} = \frac{dy}{v}$$

or, $$\frac{dx}{3x} = -\frac{dy}{3y}$$

Integrating both sides, we get:

$$\int \frac{dy}{3x} = -\int \frac{dy}{3y}$$

$$\frac{1}{3}ln(x) = -\frac{1}{3}ln(y) + \frac{1}{3}ln(C)$$

where, C is constant.

or, $$ln(xy) = ln(C) \quad \text{or} \quad xy = C$$

For the streamline passing through (2, 2),

$$C = 2 \times 2 = 4$$

Hence, the required streamline equation is: $\boldsymbol{xy = 4}$ **(Ans.)**

(*ii*)

$$V = -y^2 i - 6xj$$

$$u = -y^2 \text{ and } v = -6x$$

$$-\frac{dx}{y^2} = -\frac{dy}{6x} \quad \text{or} \quad 6x\,dx = y^2 dy$$

$$\int 6x\,dx = \int y^2\,dy$$

or,

$$\frac{6x^2}{2} = \frac{y^3}{3} + C$$

or,

$$3x^2 - \frac{y^3}{3} = C$$

Putting, $x = 2$, $y = 2$, we get:

$$3 \times (2)^2 - \frac{(2)^3}{3} = C \quad \text{or} \quad C = \frac{28}{3}$$

Hence the equations of the required streamline is:

$$3x^2 - \frac{y^3}{3} = \frac{28}{3}$$

$$\boldsymbol{9x^2 - y^3 = 28} \text{ (Ans.)}$$

Example (6): *Velocity for a two dimensional flow field is given by*

$$V = (3 + 2xy + 4t^2)\, i + (xy^2 + 3t)\, j$$

Find the velocity and acceleration at a point (1,2) *after* 2 *sec.*

8. Control Volume and System Representations

System is a collection of matter of fixed identity (always the same atoms or fluid particles), which may move, flow, and interact with its surroundings. A *control volume*, on the other hand, is a volume in space (a geometric entity, independent of mass) through which fluid may flow.

Systems of Fluid: a specific identifiable quantity of matter that may consist of a relatively large amount of mass (the earth s atmosphere) or a single fluid particle. They are always the same fluid particles which may interact with their surroundings.

Example: following a system the fluid passing through a compressor, We can apply the equations of motion to the fluid mass to describe their behavior, but in practice it is very difficult to follow a specific quantity of matter.

Control Volume: is a volume or space through which the fluid may flow, usually associated with the geometry.

We interested in determining the forces put on a fan, airplane, or automobile by air flowing past the object obtained by following a given portion of the air (a system) as it flows along. For many situations we use the control volume approach. We identify a specific volume in space (a volume associated with the fan, airplane, or automobile, for example) and analyze the fluid flow within, through, or around that volume. In general, the control volume can be a moving volume.

The matter within a control volume may change with time as the fluid flows through it. Similarly, the amount of mass within the volume may change with time. The control volume itself is a specific geometric entity, independent of the flowing fluid.

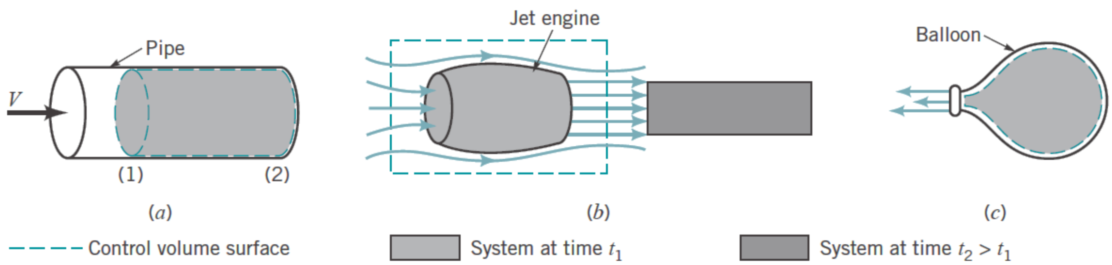
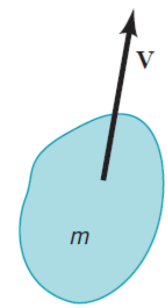


Fig. 1. Typical control volumes: (a) fixed control volume, (b) fixed or moving control volume, (c) deforming control volume.

9. Reynolds Transport Theorem

We need to describe the laws governing fluid motion using both system concepts (a given mass of the fluid) and control volume concepts (a given volume). To do this we need an analytical tool to shift from one representation to the other. The Reynolds transport theorem provides this tool.



B	$b = B/m$
m	1
$m\mathbf{V}$	$\mathbf{V}$
$\frac{1}{2}mV^2$	$\frac{1}{2}V^2$

All physical laws are stated in terms of various physical parameters. Velocity, acceleration, mass, temperature, and momentum are but a few of the more common parameters. Let *B* represent any of these fluid parameters and *b* represent the amount of that parameter per unit mass m is the mass of the portion of fluid of interest . That is,

$$\boldsymbol{B = mb}$$

B is termed an extensive property, and ***b*** is an intensive property. B is directly proportional to mass, and b is independent of mass.

Most of the laws governing fluid motion involve the time rate of change of an extensive property of a fluid system

$$\frac{dB_{\text{sys}}}{dt} = \frac{d\left(\int_{\text{sys}} \rho b \, d\mathcal{V}\right)}{dt}$$, and $$\frac{dB_{\text{cv}}}{dt} = \frac{d\left(\int_{\text{cv}} \rho b \, d\mathcal{V}\right)}{dt}$$

After derivative above equations we found Reynolds transport theorem in restricted form:

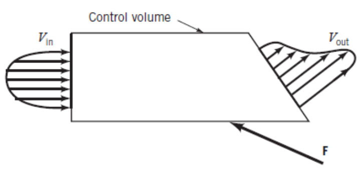
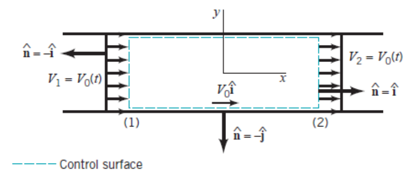
$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial B_{\text{cv}}}{\partial t} + \rho_2 A_2 V_2 b_2 - \rho_1 A_1 V_1 b_1$$

Restrictions for the above Equation:

1) Fixed control volume
2) One inlet and one outlet
3) Uniform properties
4) Normal velocity to section 1 & 2

Reynolds transport theorem (general form), This form is for a fixed non deforming control volume.

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{cv}} \rho b \, d\mathcal{V} + \int_{\text{cs}} \rho b \, \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$$

Steady flow	Unsteady Effects (inflow = outflow)
$\frac{DB_{sys}}{Dt} = \int_{cs} \rho b \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$	$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho b\, d\forall$
	

The Reynolds transport theorem for a **control volume moving with constant velocity** is given by

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{cv} \rho b\, d\forall + \int_{cs} \rho b\, \mathbf{W} \cdot \hat{\mathbf{n}}\, dA$$

10. Selection of a Control Volume

Figure below illustrates three possible control volumes associated with flow through a pipe. If the problem is to determine the pressure or velocity at point (1), the selection of the control volume (*a*) is better than that of (*b*) because point (1) lies on the control surface control surface, the values at the inlet and exit are normal to the surface. Similarly, control volume (*a*) is better than (*c*) because the flow is normal to the inlet and exit portions of the control volume.

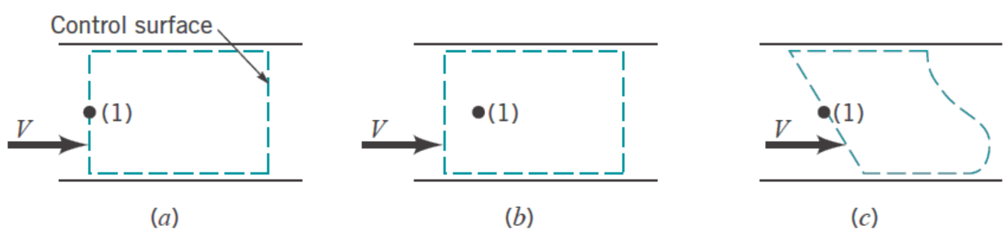
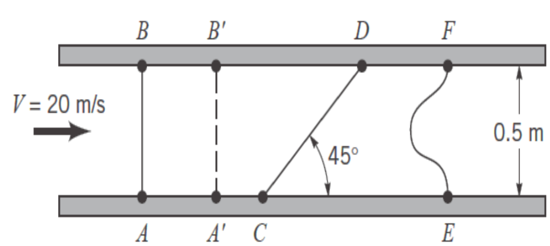


Fig. 2. Various control volumes for flow through a pipe.

Example(1) : Water flows through a duct of square cross section as shown in Fig. with a constant, uniform velocity of V=20 m/s. Consider fluid particles that lie along line A-B at time t=0. Determine the position of these particles, denoted by line A\-B\, when t=0.2 sec. Use the volume of fluid in the region between lines A–B and A\-B\, to determine the flowrate in the duct. Repeat the problem for fluid particles originally along line C–D; along line E–F. Compare your three answers.



Since V is constant in time and space, all particles on line AB move a distance $\ell = V\Delta t \doteq (20\frac{m}{s})(0.2s) = 4m$ from $t=0$ to $t=0.2s$

Thus, the volume of $ABA'B'$ is $\forall_{ABA'B'} = (0.5m)^2(4m) = 1.00\ m^3$

so that

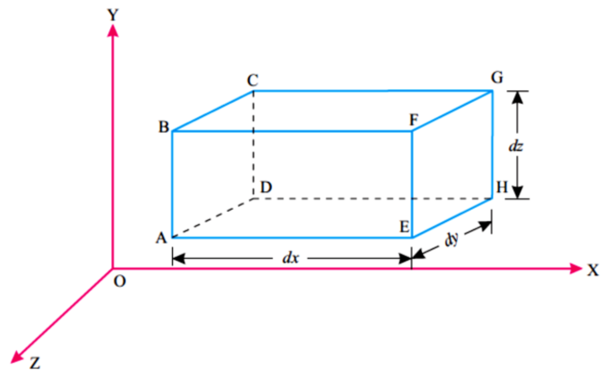
$$Q = \frac{\forall_{ABA'B'}}{\Delta t} = \frac{1.00\ m^3}{0.2s} = \underline{\underline{5.0\frac{m^3}{s}}}$$

Similarly from $t=0$ to $t=0.2s$ the fluid along lines CD and EF move to $C'D'$ and $E'F'$, respectively. Also, $\forall_{CDC'D'} = \forall_{EFE'F'} = \forall_{ABA'B'}$ so that we obtain $Q = \frac{\forall}{\Delta t} = 5.0\frac{m^3}{s}$ regardless which line we consider.

11. The Differential Equation of Mass Conservation (Continuty Equation)

The law of conservation of mass for a differential control volume requires that the net rate of change of mass within the control volume is equal to the rate at which mass flows into the control volume minus the rate at which mass flows out of the control volume. Here we choose an infinitesimal fixed control volume (*dx*, *dy*, *dz*), as in Fig.(3). The flow through each side of the element is approximately one-dimensional, and so the appropriate mass-conservation relation to use here is

$$\int_{CV} \frac{\partial \rho}{\partial t} dV = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$



The element is so small that the volume integral simply reduces to a differential term

$$\int_{CV} \frac{\partial \rho}{\partial t} dV \approx \frac{\partial \rho}{\partial t}\ dx\ dy\ dz$$

The mass-flow terms occur on all six faces, three inlets and three outlets. Where all fluid properties are considered to be uniformly varying functions of time and position:

the mass flows on the *x* (left and right faces). The flows on the *y* (bottom and top) and the *z* (back and front). We can list all these six flows as follows:

Face	Inlet mass flow	Outlet mass flow
x	$\rho u\, dy\, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dy\, dz$
y	$\rho v\, dx\, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)dy\right] dx\, dz$
z	$\rho w\, dx\, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)dz\right] dx\, dy$

Introduce these terms into Eq. above and we have

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz = \left[\rho u\, dy\, dz - \left[\rho u + \frac{\partial}{\partial x}(\rho u)dx\right] dy\, dz\right] + \left[\rho v\, dx\, dz - \left[\rho v + \frac{\partial}{\partial y}(\rho v)dy\right] dx\, dz\right] + \left[\rho w\, dx\, dy - \left[\rho w + \frac{\partial}{\partial z}(\rho w)dz\right] dx\, dy\right]$$

Many of the terms cancel each other out; after combining and simplifying the remaining terms,

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u)dx\, dy\, dz + \frac{\partial}{\partial y}(\rho v)dx\, dy\, dz + \frac{\partial}{\partial z}(\rho w)dx\, dy\, dz = 0$$

The element volume cancels out of all terms, after rearrangement we end up with the following differential equation for conservation of mass or ***Continuity equation*** in Cartesian coordinates:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

For steady flow ($\partial\rho/\partial t=0$), incompressible flow ($\rho = constant$)the equation reduce to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Example (2): *Determine which of the velocity component sets given below satisfy the equation of continuity:*

(i) $u = A \sin xy \quad , \quad v = -A \sin xy$

(ii) $u = x + y \quad , \quad v = x - y$

(iii) $u = 2x^2 + 3y \;, v = -2xy + 3y^3 + 3zy \quad , \quad w = \frac{3}{2}z^2 - 2xz - 6yz$

Solution. (*i*) $u = A \sin xy;\ v = -A \sin xy$

$$\frac{\partial u}{\partial x} = Ay \cos xy\ ;\ \frac{\partial v}{\partial y} = -Ax \cos xy$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = Ay \cos xy - Ax \cos xy \neq 0$$

i.e. Continuity equation is **not satisfied.** **(Ans.)**

(*ii*) $u = x + y;\ v = x - y$

$$\frac{\partial u}{\partial x} = 1;\ \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 1 - 1 = 0$$

i.e. Continuity equation is **satisfied** **(Ans.)**

(*iii*) $u = 2x^2 + 3y;\ v = -2xy + 3y^3 + 3zy;\ w = -\frac{3}{2} z^2 - 2xz - 6yz$

$$\frac{\partial u}{\partial x} = 4x;\ \frac{\partial v}{\partial y} = -2x + 9y^2 + 3z\ ;\ \frac{\partial w}{\partial z} = -3z - 2x - 6y$$

Hence, $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 4x - 2x + 9y^2 + 3z - 3z - 2x - 6y \neq 0$$

i.e. Continuity equation is **not satisfied** **(Ans.)**

Example (3): *In three-dimensional incompressible third flow, the velocity components in x and y-directions are:* $u = x^2 + y^2 z^3$; $v = -(xy + yz + zx)$. *Use continuity equation to evaluate an expression for the velocity component w in the z-direction.*

Solution. The continuity equation for a steady, three-dimensional incompressible fluid flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ...(i)$$

$$u = x^2 + y^2 z^3;\ v = -(xy + yz + zx)$$

$$\frac{\partial u}{\partial x} = 2x\ ;\ \frac{\partial v}{\partial y} = -(x + z)$$

Substituting these values in eqn. (*i*), we get:

$$2x - (x + z) + \frac{\partial w}{\partial z} = 0$$

or, $$\frac{\partial w}{\partial z} = -x + z$$

Integrating *w.r.t.z* we have:

$$w = -xz + \frac{z^2}{2} + C$$

where C is a constant of integration which should be independent of z but may be function of x and/or y *i.e.* $C = f(x, y)$

$$\therefore \qquad w = -x + \frac{z^2}{2} + f(x, y) \quad \textbf{(Ans.)}$$

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Example (4): *The velocity components in x and y directions are given as* $u = 2xy^3/3 - x^2y$ and $v = xy^2 - 2yx^3/3$. *Indicate whether the given velocity distribution is*:

(*i*) *A possible field of flow;*

(*ii*) *Not a possible field of flow.*

Solution. *Given.* $u = 2xy^3/3 - x^2y$, $v = xy^2 - 2yx^3/3$ ...Velocity components

A possible flow field (two-dimensional) must satisfy the continuity equation.

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad ...(i)$$

Now,
$$\frac{\partial u}{\partial x} = \frac{2}{3}y^3 - 2xy, \quad \frac{\partial v}{\partial y} = 2xy - \frac{2}{3}x^3$$

Substituting these values in eqn. (*i*), we get:

$$\left(\frac{2}{3}y^3 - 2xy\right) + \left(2xy - \frac{2}{3}x^3\right) = \frac{2}{3}(y^3 - x^3)$$

Since the continuity equation is *not* satisfied, the given velocity components, therefore, **do not represent a possible flow field. (Ans.)**

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Power Mechanics Technical Engineering Department

Second Stage

Fluid Mechanics - Dynamic

Chapter Two: Fluid Dynamics

M.Sc Zahraa A. Nadeem

Fluid Dynamics

1. Equations of Motion

Dynamics of fluid flow is the study of fluid motion with the forces causing flow. The dynamic behavior of the fluid flow is analyzed by the Newton's second law of motion, which relates the acceleration with the forces.

According to Newton's second law of motion, *the net force F_x acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a_x in the x-direction*. Thus mathematically.

$$F_x = m\,a_x$$

In the fluid flow, the following forces are present:

i. F_g, gravity force.
ii. F_p, the pressure force.
iii. F_v, force due to viscosity.
iv. F_t, force due to turbulence.
v. F_c, force due to compressibility.

Thus, the net force

$$F_x = \left(F_g\right)_x + \left(F_p\right)_x + (F_v)_x + (F_t)_x + (F_c)_x$$

a) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = \left(F_g\right)_x + \left(F_p\right)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion**,

b) For flow, where F_t is negligible, the resulting equations of motion are known as **Navier Stokes Equation**.

c) If the flow is assumed to be ideal, viscous force *(F_v)* is zero and equation of motions are known as **Euler's equation of motion**.

2. Euler's Equation of Motion

This is equation of motion in which the forces due to **gravity and pressure** are taken into consideration. This is derived by considering the motion of a fluid element along a stream-line as shown in *Fig.* Consider a cylindrical element of cross-section dA and length ds. The forces acting on the cylindrical element are:

1. *Pressure force* $\boldsymbol{pdA}$ *in the direction of flow.*
2. *Pressure force* $\left(\boldsymbol{p+\frac{\partial p}{\partial s}ds}\right)\boldsymbol{dA}$ *opposite to the direction of flow.*
3. *Weight of element* $\boldsymbol{\rho gdAds}$.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element multiplied by acceleration in the direction s. and after arrangements

$$\frac{1}{\rho}\frac{\partial p}{\partial s}+g\,\frac{dz}{ds}+v\frac{\partial v}{\partial s}=0$$

$$\boxed{\boldsymbol{\frac{dp}{\rho}+g\,dz+vdv=0}}$$

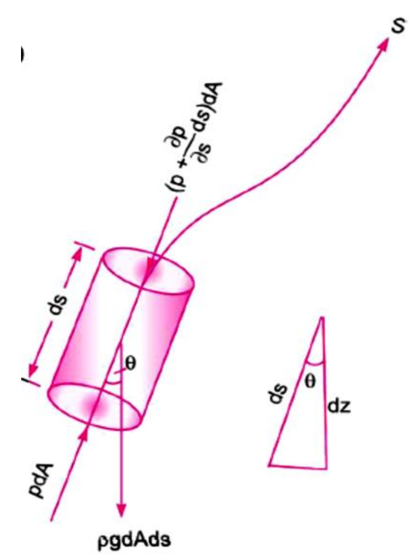
This equation is known as ***Euler's equation of motion.***

3. Bernoulli's Equation

Bernoulli's equation is *an approximate relation between pressure, velocity, and elevation*.

The assumptions made in the derivation of Bernoulli's equation:

i. *The fluid is ideal, viscosity is zero*
ii. *The flow is steady*
iii. *The flow is incompressible*
iv. *The flow is irrotational.*



Bernoulli equation is obtained by integrating the *Euler's equation of motion as:*

$$\int\frac{dp}{\rho}+\int g\,dz+\int vdv=constant$$

If flow is incompressible, ρ is constant and

$$\frac{p}{\rho} + gz + \frac{v^2}{2} = constant \quad or \quad \frac{p}{\rho g} + z + \frac{v^2}{2g} = constant$$

$$\boxed{\boldsymbol{\frac{p}{\rho g} + \frac{v^2}{2g} + z = constant}}$$

This equation is ***Bernoulli's equation*** in which

$\frac{p}{\rho g} = pressure\ energy\ per\ unit\ weight\ of\ fluid\ or\ \boldsymbol{pressure\ head};$

$(it\ represents\ the\ height\ of\ a\ fluid\ column\ that\ produces\ the\ static\ pressure\ P)$

$\frac{v^2}{2g} = kinetic\ energy\ per\ unit\ weight\ or\ \boldsymbol{kinetic\ head}; (it\ represents\ the$
$elevation\ needed\ for\ a\ fluid\ to\ reach\ the\ velocity\ V\ during\ frictionless\ free\ fall)$

$z = potential\ energy\ per\ unit\ weight\ or\ \boldsymbol{evelation\ head.}$

The summation of these three heads represent **Total Head** (*H*).

The Bernoulli equation can also be written between any two points on the same streamline as

$$\frac{(p_2 - p_1)}{\rho} + \frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) = constant$$

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = constant$$

4. Static, Dynamic, and Stagnation Pressures

If multiplying the Bernoulli equation by the densityρ, and g we obtained

$$P + \rho\frac{V^2}{2} + \rho gz = constant$$

Each term in this equation has pressure units, and thus each term represents some kind of pressure:

- P is the **static pressure** (it does not incorporate any dynamic effects).
- $\rho V^2/2$ is the **dynamic pressure;** it represents the pressure rise when the fluid in motion.
- ρgz is the **hydrostatic pressure,** which is not pressure in a real sense since its value depends on the reference level selected; it accounts for the elevation effects.

The sum of the static, dynamic, and hydrostatic pressures is called the **total pressure.** Therefore, the Bernoulli equation states that *the total pressure along a streamline is constant*.

The sum of the static and dynamic pressures is called the **stagnation pressure,** and it is expressed as

$$P_{stag} = P + \rho\frac{V^2}{2}$$

The static, dynamic, and stagnation pressures are shown in Fig. 1.

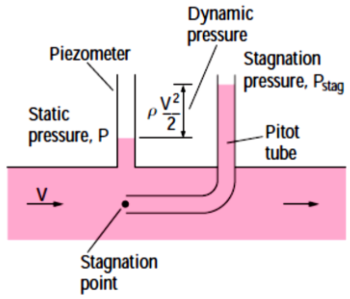


Fig.1 The static, dynamic, and stagnation pressures.

When static and stagnation pressures are measured at a specified location, the fluid velocity at that location can be calculated from

$$V = \sqrt{\frac{2(P_{stag} - P)}{\rho}}$$

5. Pitot Tube

Pitot tube *is one of the most accurate devices for velocity measurement.* It works on the *principle that if the velocity of flow at a point becomes zero, the pressure there is increased due to conversion of kinetic energy into pressure.*

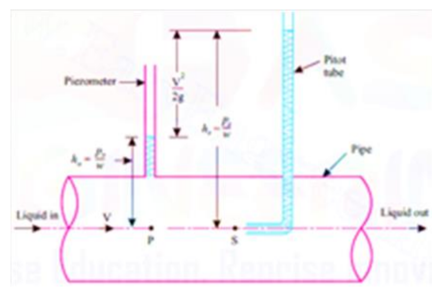


Fig.2 Pitot Tube

Applying Bernoulli's equation between stagnation point (S) and point (P) in the undisturbed flow at the same horizontal plane, we get:

$$\frac{p_0}{w} + \frac{V^2}{2g} = \frac{p_s}{w} \quad \text{or} \quad h_0 + \frac{V^2}{2g} = h_s$$

or,

$$V = \sqrt{2g\,(h_s - h_0)} \quad \text{or} \quad \sqrt{2g\,\Delta h} \qquad \text{...(1)}$$

where,

p_0 = Pressure at point 'P', *i.e.* static pressure,
V = Velocity at point 'P', *i.e.* free flow velocity,
p_s = Stagnation pressure at point 'S', and
Δh = Dynamic pressure
= Difference between stagnation pressure head (h_s) and static pressure head (h_0).

where C_v = Co-efficient of pitot-tube

$\therefore$ Velocity at any point $\quad v = C_v\sqrt{2gh}$

6. Value of 'h' given by differential U-tube manometer

Case I: Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let,

S_h = Sp. gravity of the heavier liquid
S_o = Sp. gravity of the liquid flowing through pipe
x = Difference of the heavier liquid column in U-tube

$$h = x\left[\frac{S_h}{S_o} - 1\right]$$

Case II: If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$

where S_l = Sp. gr. of lighter liquid in U-tube
S_o = Sp. gr. of fluid flowing through pipe
x = Difference of the lighter liquid columns in U-tube.

Example (1): *Petroleum oil (sp. gr. = 0.9 and viscosity = 13 cP) flows isothermally through a horizontal 5 cm pipe. A Pitot tube is inserted at the centre of a pipe and its leads are filled with the same oil and attached to a U-tube containing mercury(sp. gr. = 13.6). The reading on the manometer is 10 cm. Calculate the volumetric flow of oil in m^3/s. The co-efficient of Pitot tube is 0.98.*

Solution. *Given*: Sp gr. of oil = 0.9; $\mu = 13 \text{ cP} = \frac{13}{100} \times 0.1 \text{ Ns/m}^2 = 0.013 \text{ Ns/m}^2$;

y = 10 cm of Hg = 0.1 m of Hg., D = 5 cm = 0.05 m;

Co-efficient of Pitot tube, $C_v = 0.98$

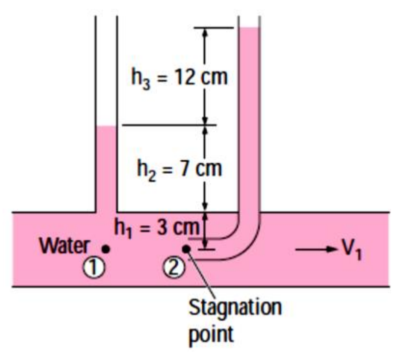
Volumetric flow of oil:

Differential head, $h = y\left(\frac{S_{Hg}}{S_{Oil}} - 1\right) = 0.1\left(\frac{13.6}{0.9} - 1\right) = 1.411$

$\therefore$ Actual velocity of flow, $V = C_v\sqrt{2gh} = 0.98\sqrt{2 \times 9.81 \times 1.411} = \mathbf{5.156 \text{ m/s}}$

Volumetric flow of oil $= A \times V = \frac{\pi}{4} \times 0.05^2 \times 5.156 = \mathbf{0.01 \text{ m}^3\text{/s (Ans.)}}$

Example (2): *A piezometer and a Pitot tube are tapped into a horizontal water pipe, as shown in Fig., to measure static and stagnation pressures. For the indicated water column heights, determine the velocity at the center of the pipe.*



Solution:

$$P_1 = \rho g(h_1 + h_2)$$

$$P_2 = \rho g(h_1 + h_2 + h_3)$$

Noting that point 2 is a stagnation point and thus $V_2 = 0$ and $z_1 = z_2$, the application of the Bernoulli equation between points 1 and 2 gives

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + \cancel{z_1} = \frac{P_2}{\rho g} + \frac{V_2^2}{2g}^{\nearrow 0} + \cancel{z_2} \quad \rightarrow \quad \frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g}$$

Substituting the P_1 and P_2 expressions gives

$$\frac{V_1^2}{2g} = \frac{P_2 - P_1}{\rho g} = \frac{\rho g(h_1 + h_2 + h_3) - \rho g(h_1 + h_2)}{\rho g} = h_3$$

Solving for V_1 and substituting,

$$V_1 = \sqrt{2gh_3} = \sqrt{2(9.81\ \text{m/s}^2)(0.12\ \text{m})} = 1.53\ \text{m/s}$$

7. Hydraulic Grade Line (HGL) and Energy Grade Line (EGL)

If a piezometer (measures static pressure) is tapped into a pipe, as shown in Fig. 3, the liquid would rise to a height of $p/\rho g$ above the pipe center. The *hydraulic grade line* (HGL) is obtained by doing this at several locations along the pipe and drawing a line through the liquid levels in the piezometers. The vertical distance above the pipe center is a measure of pressure within the pipe.

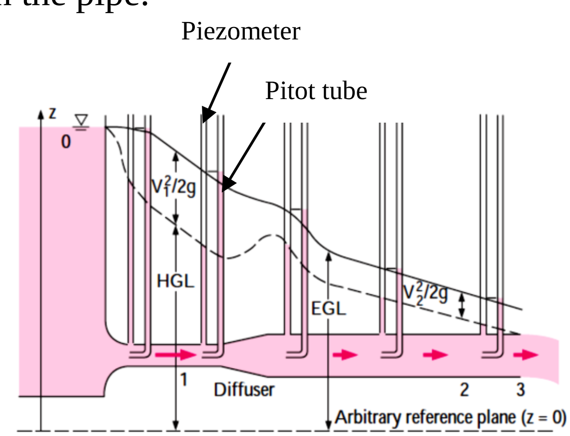


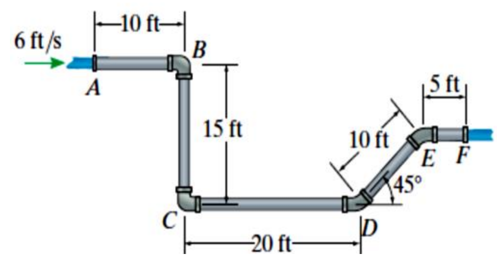
Fig.3 The hydraulic grade line and the energy grade line for free discharge from a reservoir through a horizontal pipe.

Similarly, if a Pitot tube (measures static + dynamic pressure) is tapped into a pipe, the liquid would rise to a height of $p/\rho g + V^2/2g$ above the pipe center, or a distance of

$V^2/2g$ above the HGL. The *energy grade line* (EGL) is obtained by doing this at several locations along the pipe and drawing a line through the liquid levels in the Pitot tubes.

Noting that the fluid also has elevation head z, the HGL and EGL can be defined as follows: The line that represents the sum of the static pressure and the elevation heads, $p/\rho g + z$, is called the **hydraulic grade line.** The line that represents the total head of the fluid, $p/\rho g + V^2/2g + z$, is called the **energy grade line.** The difference between the heights of EGL and HGL is equal to the dynamic head, $V^2/2g$.

Example (3): *Water flows through the constant diameter pipe such that at A it has a velocity of 6 ft/s and a pressure of 30 psi. Draw the energy and hydraulic grade lines for the flow from A to F using a datum through CD.*



Solution:

EL and HGL. With reference to the datum through *CD*, $z_A = 15$ ft. Thus, *EL* will have a constant value of

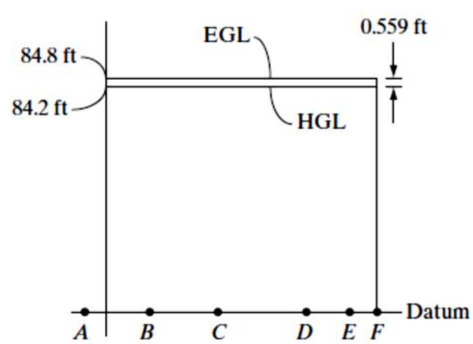
$$H = \frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A$$

$$= \frac{\left(30\frac{\text{lb}}{\text{in}^2}\right)\left(\frac{12\text{ in.}}{1\text{ ft}}\right)^2}{62.4\text{ lb/ft}^3} + \frac{(6\text{ ft/s})^2}{2(32.2\text{ ft/s}^2)} + 15\text{ ft} = 84.8\text{ ft}$$

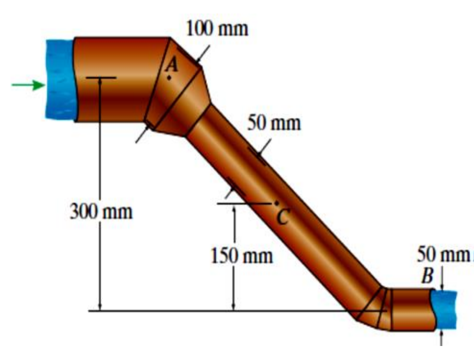
Since the pipe has a constant diameter, the velocity head will have a constant value of

$$\frac{V_A^2}{2g} = \frac{(6\text{ ft/s})^2}{2(32.2\text{ ft/s}^2)} = 0.559\text{ ft}$$

A plot of the *EL* and *HGL* is shown in Fig.



Example (4): *Water at a pressure of 80 kPa and a velocity of 2 m/s at A flows through the transition. Determine the velocity and the pressure at B. Draw the energy and hydraulic grade lines using a datum at B.*



Solution:

Continuity Equation.

$$V_A A_A = V_B A_B$$

$$2 \times \frac{\pi}{4} 0.1^2 = V_B \times \frac{\pi}{4} 0.05^2$$

$$\boldsymbol{V_B = 8\frac{m}{s}}$$

Bernoulli Equation. With reference to the datum through B, $z_A = 0.3$ m and $z_B = 0$.

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B$$

$$\frac{80(10^3)\frac{\text{N}}{\text{m}^2}}{(1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)} + \frac{(2\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} + 0.3\ \text{m}$$

$$= \frac{p_B}{(1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)} + \frac{(8\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} + 0$$

$p_B = 52.943(10^3)\ \text{Pa} = 52.9\ \text{kPa}$ **Ans.**

EL and HGL. *EL* will have a constant value of

$$H = \frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A$$

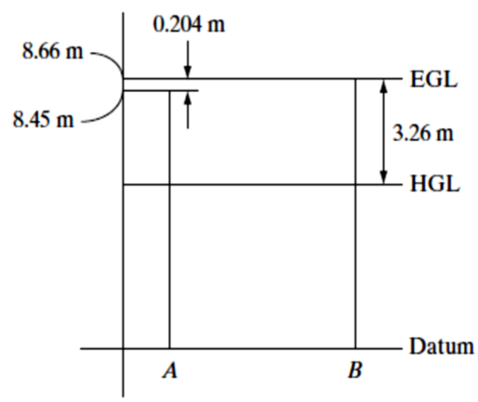
$$= \frac{80(10^3)\frac{\text{N}}{\text{m}^2}}{(1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)} + \frac{(2\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} + 0.3\ \text{m} = 8.66\ \text{m}$$

The velocity head before A is

$$\frac{V_A^2}{2g} = \frac{(2\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} = 0.204\ \text{m}$$

The velocity head after A is

$$\frac{V_A^2}{2g} = \frac{(8\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} = 3.26\ \text{m}$$



The *EL* and *HGL* are plotted as shown in Fig.

8. Orifice Meter

It is a device used for measuring the rate of flow of a fluid flowing through a pipe. It consists of flat circular plate which has a circular sharp edged hole, in concentric with the pipe, this is called **orifice**. The diameter of orifice is generally 0.5 times the diameter of the pipe, although it may vary from 0.4 to 0.8 times the pipe diameter.

The orifice plate is one in a group known as differential pressure flow meters, in simple terms the pipeline fluid is passed through a restriction, and the pressure differential is measured across that restriction. Based on the work of Bernoulli, the relationship between the velocity of fluid passing through the orifice is proportional to the square root of the pressure loss across it.

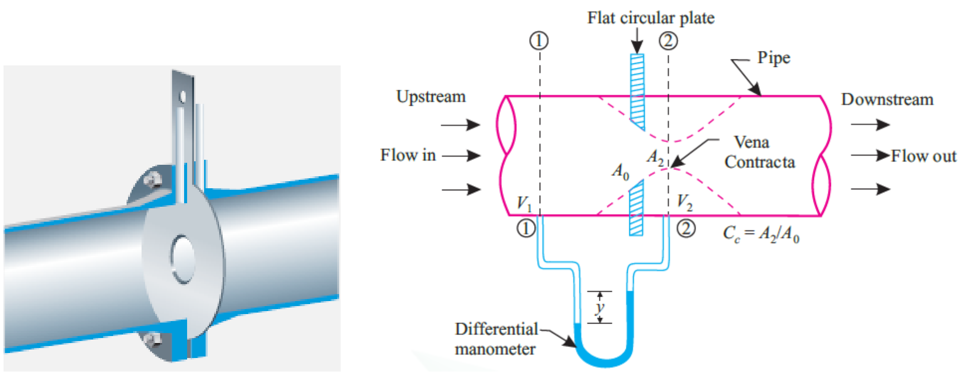


Fig.(4) Orifice meter

The discharge is:

$$Q = C_d \frac{A_0 \cdot A_1 \sqrt{2gh}}{\sqrt{A_1^2 - A_0^2}}$$

Where C_d is Coefficient of discharge.

<u>Example (5):</u> *An orifice meter with orifice diameter 15 cm is inserted in a pipe of 30 cm diameter. The pressure difference measured by a mercury oil differential manometer on the two sides of the orifice meter gives a reading of 50 cm of mercury. Find the rate of flow of oil of sp. gr. 0.9 when the coefficient of discharge of the orifice meter = 0.64. sp. gr. for mercury is 13.6.*

<u>Solution:</u>

Dia. of orifice, $d_0 = 15$ cm

$\therefore$ Area, $a_0 = \frac{\pi}{4}(15)^2 = 176.7 \text{ cm}^2$

Dia. of pipe, $d_1 = 30$ cm

$\therefore$ Area, $a_1 = \frac{\pi}{4}(30)^2 = 706.85 \text{ cm}^2$

Sp. gr. of oil, $S_o = 0.9$

Reading of diff. manometer, $x = 50$ cm of mercury

$\therefore$ **Differential head,** $$h = x\left[\frac{S_g}{S_o} - 1\right] = 50\left[\frac{13.6}{0.9} - 1\right] \text{ cm of oil}$$

$$= 50 \times 14.11 = 705.5 \text{ cm of oil}$$

$$C_d = 0.64$$

$$Q = C_d \cdot \frac{a_0 a_1}{\sqrt{a_1^2 - a_0^2}} \times \sqrt{2gh}$$

$$= 0.64 \times \frac{176.7 \times 706.85}{\sqrt{(706.85)^2 - (176.7)^2}} \times \sqrt{2 \times 981 \times 705.5}$$

$$= \frac{94046317.78}{684.4} = 137414.25 \text{ cm}^3/\text{s} = \mathbf{137.414 \text{ litres/s. Ans.}}$$

Example (6): *Water flows at the rate of 0.015m^3/s through a 100 mm diameter orifice used in a 200mm pipe. What is the difference of pressure head between the upstream section and the vena contracta section? Take co-efficient of contraction C_c=0.60 and C_v=1.0.*

Solution. *Given*: $Q = 0.015$ m^3/s; $D_0 = 100$ mm $= 0.1$ m; $D_1 = 200$ mm $= 0.2$ m; $C_c = 0.60$; $C_v = 1.0$

Difference in pressure head h: Refer to Fig. 6.35.

$$A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4} \times 0.2^2 = 0.03142 \text{ m}^2$$

$$A_0 = \frac{\pi}{4}D_0^2 = \frac{\pi}{4} \times 0.1^2 = 0.007854 \text{ m}^2$$

$$C_d = C_c \times C_v = 0.60 \times 1.0 = 0.6$$

Using the relation: $$Q = C_d \frac{A_0 A_1 \sqrt{2gh}}{\sqrt{A_1^2 - A_0^2}}$$

or, $$0.015 = 0.6 \times \frac{0.007854 \times 0.03142\sqrt{2 \times 9.81 \times h}}{\sqrt{(0.03142)^2 - (0.007854)^2}} \quad \text{...[Eqn. (6.9)]}$$

or, $$0.015 = 0.6 \times \frac{0.001093\sqrt{h}}{0.03042}$$

or, $$h = \left(\frac{0.015 \times 0.03042}{0.6 \times 0.001093}\right)^2 = \mathbf{0.484 \text{ m of water (Ans.)}}$$

9. Venturimeter

A venturimeter is a device used for measuring the rate of a flow of a fluid flowing through a pipe. It is based on the principle of Bernoulli's equation. The Venturimeter has a converging conical section from the initial pipe diameter (inlet), followed by a throat, and ended with a diverging conical section (outlet) back to the original pipe diameter.

Types of venturimeters:

Venturimeters may be *classified* as follows:

1. Horizontal venturimeters.
2. Vertical venturimeters.
3. Inclined venturimeters.

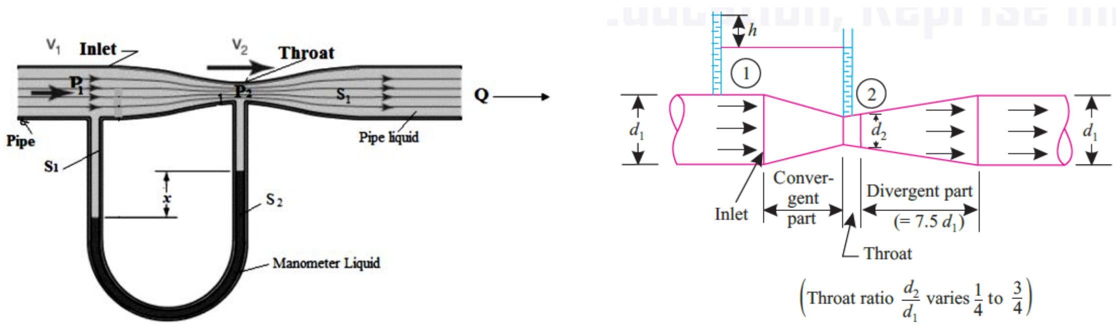


Fig. 5. Venturimeter.

As the inlet area of the venturimeter is larger than the throat area, the velocity at the throat increases resulting in decrease of pressure. By this, a pressure difference is created between the inlet and the throat of the venturimeter. The pressure difference is measured by using a differential U-tube manometer. This pressure difference helps in the determination of rate of flow of fluid or discharge through the pipe line.

$$Q_{act} = C_d \times \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}$$

Where C_d is Coefficient of discharge.

Difference between a venturimeter and an orificemeter:

The main points of *difference* between a venturimeter and orificemeter are:

1. The venturimeter can be used for *measuring the flow rates of all incompressible flows.* (gases with low pressure variations, as well as liquids), whereas orificemeters are generally used for measuring the *flow rates of liquids.*

2. Venturimeter is installed *in pipeline only*, and the accelerated flow through the apparatus, is subsequently decelerated to the original velocity at the outlet of the venturimeter. The flow continues through the pipeline. In the orificemeter the entire potential energy of the fluid is converted to kinetic energy, and the jet *discharges freely into the open atmosphere.*

3. In venturimeter, the flow velocity is *measured by noting the pressure difference between the inlet and the throat of the venturimeter*, whereas in the orificemeter the discharge velocity is measured by using *Pitot tube.*

Example (7): *A horizontal venturimeter with inlet diameter 200 mm and throat diameter 100 mm is used to measure the flow of water. The pressure at inlet is 0.18 N/mm² and the vacuum pressure at the throat is 280 mm of mercury. Find the rate of flow. The value of Cd may be taken as 0.98.*

Solution. Inlet diameter of venturimeter, $D_1 = 200 \text{ mm} = 0.2 \text{ m}$

$\therefore$ Area of inlet, $A_1 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$

Throat diameter, $D_2 = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore$ Area of throat, $A_2 = \frac{\pi}{4} \times 0.1^2 = 0.00785 \text{ m}^2$

Pressure at inlet, $p_1 = 0.18 \text{ N/mm}^2 = 180 \text{ kN/m}^2$

$$\therefore \quad \frac{p_1}{w} = \frac{180}{9.81} = 18.3 \text{ m}$$

Vacuum pressure at the throat,

$$\frac{p_2}{w} = -280 \text{ mm of mercury}$$

$$= -0.28 \text{ m of mercury} = -0.28 \times 13.6 = -3.8 \text{ m of water}$$

Co-efficient of discharge, $C_d = 0.98$

$$\therefore \quad \text{Differential head, } h = \frac{p_1}{w} - \frac{p_2}{w} = 18.3 - (-3.8) = 22.1 \text{ m}$$

Rate of flow, Q:

Using the relation,

$$Q = C_d \times \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}, \text{ we have:}$$

$$= 0.98 \times \frac{0.0314 \times 0.00785}{\sqrt{(0.0314)^2 - (0.00785)^2}} \times \sqrt{2 \times 9.81 \times 22.1}$$

$$= \frac{0.000241}{0.0304} \times 20.82$$

or $\quad Q = \mathbf{0.165 \text{ m}^3/\text{s}}$ **(Ans.)**

Example (8): *A horizontal venturimeter with inlet diameter 200 mm and throat diameter 100 mm is employed to measure the flow of water. The reading of the differential manometer connected to the inlet is 180 mm of mercury. If the co-efficient of discharge is 0.98, determine the rate of flow.*

Solution. Inlet diameter of venturimeter, $D_1 = 200 \text{ mm} = 0.2 \text{ m}$

$\therefore$ Area at inlet, $A_1 = \frac{\pi}{4} \times 0.2^2 = 0.0314 \text{ m}^2$

Throat diameter, $D_2 = 100 \text{ mm} = 0.1 \text{ m}$

$\therefore$ Area of throat, $A_2 = \frac{\pi}{4} \times 0.1^2 = 0.00785 \text{ m}^2$

Reading of differential manometer, $y = 180$ mm (= 0.18 m) of mercury

Co-efficient of discharge, $C_d = 0.98$

Rate of flow, Q:

To find difference of pressure head (h) using the relation,

$$h = \left[\frac{S_{hl}}{S_p} - 1\right], \text{ we have:}$$

where, S_{hl} = Sp.gr. of mercury (heavy liquid) = 13.6, and

S_p = Sp. gr. of liquid through the pipe *i.e.*, water = 1

$$h = 0.18\left[\frac{13.6}{1} - 1\right] = 2.268 \text{ m}$$

To find Q, using the relation,

$$Q = Cd\frac{A_1A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}, \text{ we get:}$$

$$Q = 0.98 \times \frac{0.0314 \times 0.00785}{\sqrt{(0.0314)^2 - (0.00785)^2}} \times \sqrt{2 \times 9.81 \times 2.268}$$

or

$$Q = \frac{0.000241}{0.0304} \times 6.67 = \mathbf{0.0528\ m^3/s}\ \text{(Ans.)}$$

Example (9): *A 300 mm× 150 mm venturimeter is provided in a vertical pipe line carrying oil of specific gravity 0.9, the flow being upwards. The difference in elevation of throat section and entrance section of the venturimeter is 300 mm. The differential U-tube mercury manometer shows a gauge deflection of 250 mm. Calculate:*

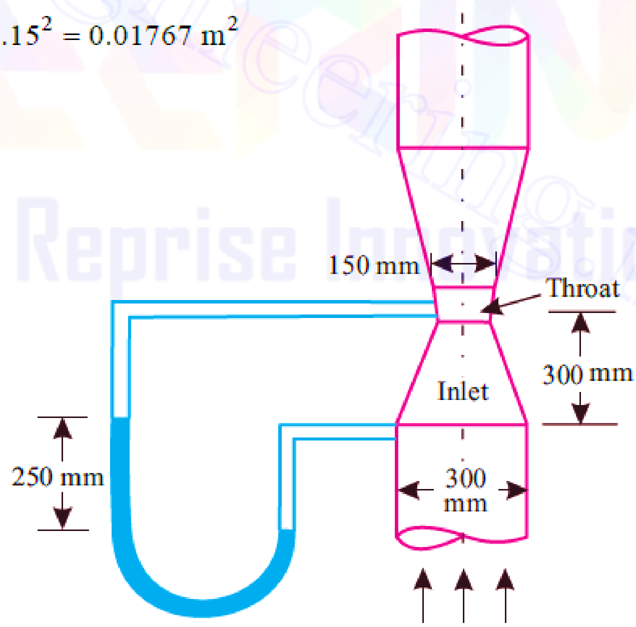
i. *The discharge of oil, and*
ii. *The pressure difference between the entrance section and the throat section. Take the coefficient of discharge as 0.98 and specific gravity of mercury as 13.6.*

Solution. Diameter at inlet, D_1 = 300 mm = 0.3 m

$\therefore$ Area of inlet, $A_1 = \frac{\pi}{4} \times 0.3^2 = 0.07 \text{ m}^2$

Diameter at throat, D_2 = 150 mm = 0.15 m

$\therefore$ Area at throat, $A_2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$



Specific gravity of heavy liquid (mercury) in U-tube manometer, S_{hl} =13.6

Specific gravity of liquid (oil) flowing through pipe, S_p = 0.9

Reading of differential manometer, y = 250 mm = 0.25 m

The differential 'h' is given by:

$$h = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)$$

$$= y\left[\frac{S_{hl}}{S_p} - 1\right] = 0.25\left[\frac{13.6}{0.9} - 1\right]$$

= 3.53 m of oil

(i) Discharge of oil, Q:

Using the relation,

$$Q = C_d \times \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}\text{, we have:}$$

$$Q = 0.98 \times \frac{0.07 \times 0.01767}{\sqrt{0.07^2 - 0.01767^2}} \times \sqrt{2 \times 9.81 \times 3.53}$$

$$= \frac{0.001212}{0.0677} \times 8.32 = \mathbf{0.1489\ m^3/s.}\ \textbf{(Ans.)}$$

(ii) Pressure difference between entrance and throat sections, $p_1 - p_2$:

We know that, $$h = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) = 3.53$$

or, $$\left(\frac{p_1}{w} - \frac{p_2}{w}\right) + (z_1 - z_2) = 3.53$$

But, $$z_2 - z_1 = 300 \text{ mm or } 0.3 \text{ m}$$

$$\therefore \quad \left(\frac{p_1}{w} - \frac{p_2}{w}\right) - 0.3 = 3.53 \quad \text{or} \quad \frac{p_1 - p_2}{w} = 3.83$$

or, $$p_1 - p_2 = (9.81 \times 0.9) \times 3.83 = \mathbf{33.8\ kN/m^2}\ \textbf{(Ans.)}$$

Example (10): *A vertical venturimeter carries a liquid of relative density 0.8 and has inlet and throat diameters of 150 mm and 75 mm respectively. The pressure connection at the throat is 150 mm above that at the inlet. If the actual rate of flow is 40 litres/sec and the Cd = 0.96, calculate the pressure difference between inlet and throat in N/m².*

Solution. *Given*: Sp. gravity = 0.8, D_1 = 150 mm = 0.15 m; D_2 = 75 mm = 0.075 m; $z_2 - z_1$ = 150 mm = 0.15 m, Q_{act} = 40 litres/sec. = 0.04 m³/s, C_d = 0.96.

Pressure difference ($p_1 - p_2$):

$$A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} \times 0.15^2 = 0.01767 \text{ m}^2$$

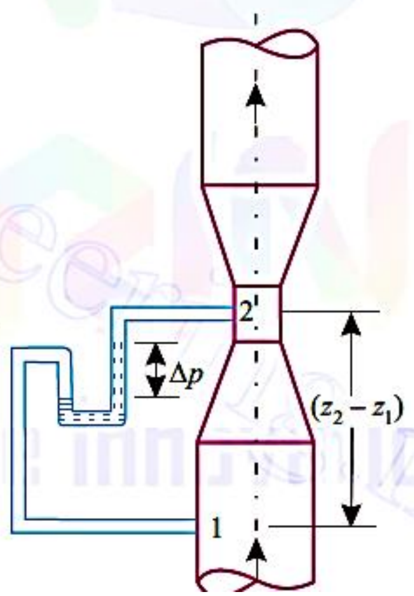
$$A_2 = \frac{\pi}{4} D_2^2 = \frac{\pi}{4} \times (0.075)^2 = 0.00442 \text{ m}^2$$

$$Q_{act} = C_d \times \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh}\text{, we get:}$$

$$0.04 = 0.96 \times \frac{0.01767 \times 0.00442}{\sqrt{0.01767^2 - 0.00442^2}} \times \sqrt{2 \times 9.81} \times \sqrt{h}$$

or, $$0.04 = 0.96 \times 0.004565 \times 4.429\sqrt{h}$$

$$\therefore \quad h = \left(\frac{0.04}{0.96 \times 0.004565 \times 4.429}\right)^2 = 4.247 \text{ m}$$



Also, $h = \left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right)$

or, $4.247 = \left(\frac{p_1}{w} - \frac{p_2}{w}\right) + (z_1 - z_2)$

$= \left(\frac{p_1 - p_2}{\rho g}\right) - 0.15$

or, $(p_1 - p_2) = \rho g\,(4.247 + 0.15)$

$= (0.8 \times 1000 \times 9.81)\,(4.247 + 0.15)\ \text{N/m}^2$

$= \mathbf{34.51\ kN/m^2}$ **(Ans.)**

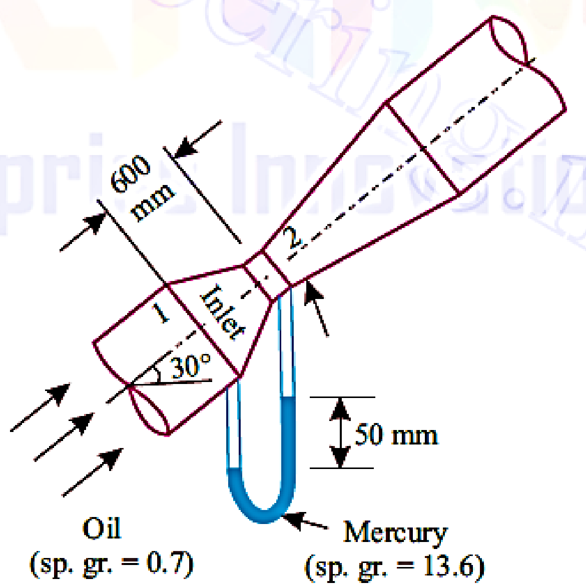
Example (11): *The following data relate to an inclined venturimeter:*
Diameter of the pipeline = 400 mm
Inclination of the pipeline with the horizontal = 30°
Throat diameter = 200 mm
The distance between the mouth and throat of the meter = 600 mm
Sp. gravity of oil flowing through the pipeline = 0.7
Sp. gravity of heavy liquid (U-tube) = 13.6
Reading of the differential manometer = 50 mm
The co-efficient of the meter = 0.98
Determine the rate of flow in the pipeline.

Solution. Diameter at inlet, $D_1 = 400\ \text{mm} = 0.4\ \text{m}$

$\therefore$ Area of inlet, $A_1 = \frac{\pi}{4} \times 0.4^2 = 0.1257\ \text{m}^2$

Throat diameter, $D_2 = 200\ \text{mm} = 0.2\ \text{m}$

$\therefore$ Area at throat, $A_2 = \frac{\pi}{4} \times 0.02^2 = 0.0314\ \text{m}^2$



Reading of the differential manometer (U-tube), $y = 50\ \text{mm} = 0.05\ \text{m}$

Difference of pressure head h is given by:

$$h = y\left[\frac{S_{hl}}{S_p} - 1\right]$$

where, S_{hl} = Sp. gravity of heavy liquid (*i.e.*, mercury) in U-tube = 13.6, and

S_p = Sp. gravity of liquid (*i.e.*, oil) flowing through the pipe = 0.7

$$\therefore \qquad h = 0.05\left(\frac{13.6}{0.7} - 1\right) = 0.92\ \text{m of oil}$$

Now, applying Bernoulli's equation at sections '1' and '2', we get:

$$\frac{p_1}{w} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{w} + z_2 + \frac{V_2^2}{2g} \qquad ...(i)$$

or, $$\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = 0$$

But, $$\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) = h$$

or, $$\left(\frac{p_1}{w} - \frac{p_2}{w}\right) + (z_1 - z_2) = h$$

It may be noted that *differential gauge reading will include in itself the difference of pressure head and the difference of datum head.*

Thus, eqn. (*i*) reduces to:

$$h + \frac{V_1^2}{2g} - \frac{V_2^2}{2g} = 0 \qquad ...(ii)$$

Applying continuity equation at sections '1' and '2' we get:

$$A_1V_1 = A_2V_2$$

or, $$V_1 = \frac{A_2V_2}{A_1} = \frac{(\pi/4)\times 0.2^2}{(\pi/4)\times 0.4^2} \times V_2 = \frac{V_2}{4}$$

Substituting the value of V_1 and h in eqn. (*ii*), we get:

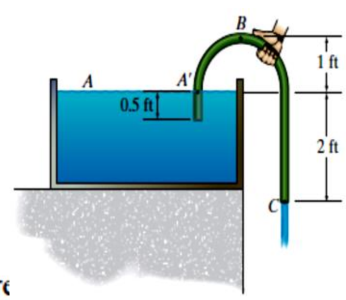
$$0.92 + \frac{V_2^2}{16\times 2g} - \frac{V_2^2}{2g} = 0$$

or, $$\frac{V_2^2}{2g}\left(1 - \frac{1}{16}\right) = 0.92 \quad \text{or} \quad V_2^2 \times \frac{15}{16} = 0.92$$

or, $$V_2^2 = \frac{0.92\times 2\times 9.81\times 16}{15} = 19.25 \quad \text{or} \quad V_2 = 4.38 \text{ m/s}$$

Rate of flow of oil, $Q = A_2V_2 = 0.0314 \times 4.38 = \mathbf{0.1375\ m^3/s}$ **(Ans.)**

Example (12): The hose is used to siphon water from the tank. Determine the smallest pressure in the tube and the volumetric discharge at *C*. The hose has an inner diameter of 0.75 in. Draw the energy and hydraulic grade lines for the hose using a datum at *C*.



Solution:

Bernoulli Equation. Since the tank is a large source, $V_A \cong 0$. Here, A and C are exposed to the atmosphere, $p_A = p_C = 0$. If the datum coincides with the horizontal line through C, $z_A = 2$ ft, $z_B = 3$ ft, and $z_C = 0$. From A to C,

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_C}{\gamma} + \frac{V_C^2}{2g} + z_C$$

$$0 + 0 + 2\text{ ft} = 0 + \frac{V_C^2}{2(32.2\text{ ft/s}^2)} + 0$$

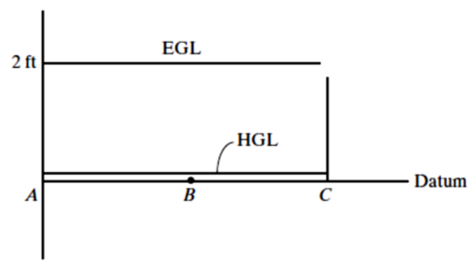
$$V_C = 11.35\text{ ft/s}$$

The smallest pressure occurs at the maximum height of the hose, point *B*. Since the hose has a constant diameter, $V_B = V_C = 11.35$ ft/s. From *A* to *B*,

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B$$

$$0 + 0 + 2\text{ ft} = \frac{p_B}{62.4\text{ lb/ft}^3} + \frac{(11.35\text{ ft/s})^2}{2(32.2\text{ ft/s}^2)} + 3\text{ ft}$$

$$p_B = \left(-187.2\frac{\text{lb}}{\text{ft}^2}\right)\left(\frac{1\text{ ft}}{12\text{ in.}}\right)^2 = -1.30\text{ psi}$$



Discharge.

$$Q = V_C A_C = (11.35\text{ ft/s})\left[\pi\left(\frac{0.375}{12}\text{ ft}\right)^2\right]$$
$$= 0.0348\text{ ft}^3/\text{s}$$

$$\frac{V_B^2}{2g} = \frac{V_C^2}{2g} = 2\text{ ft}$$

EXAMPLE 3.10 Siphon and Cavitation

GIVEN A liquid can be siphoned from a container as shown in Fig. E3.10*a* provided the end of the tube, point (3), is below the free surface in the container, point (1), and the maximum elevation of the tube, point (2), is "not too great." Consider water at 60° F being siphoned from a large tank through a constant diameter hose as shown in Fig. E3.10*b*. The end of the siphon is 5 ft below the bottom of the tank, and the atmospheric pressure is 14.7 psia.

FIND Determine the maximum height of the hill, *H*, over which the water can be siphoned without cavitation occurring.

SOLUTION

If the flow is steady, inviscid, and incompressible we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as follows:

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3 \qquad \textbf{(1)}$$

With the tank bottom as the datum, we have $z_1 = 15$ ft, $z_2 = H$, and $z_3 = -5$ ft. Also, $V_1 = 0$ (large tank), $p_1 = 0$ (open tank), $p_3 = 0$ (free jet), and from the continuity equation $A_2V_2 = A_3V_3$, or because the hose is constant diameter, $V_2 = V_3$. Thus, the speed of the fluid in the hose is determined from Eq. 1 to be

$$V_3 = \sqrt{2g(z_1 - z_3)} = \sqrt{2(32.2\text{ ft/s}^2)[15 - (-5)]\text{ ft}}$$
$$= 35.9\text{ ft/s} = V_2$$

Use of Eq. 1 between points (1) and (2) then gives the pressure p_2 at the top of the hill as

$$p_2 = p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 - \tfrac{1}{2}\rho V_2^2 - \gamma z_2$$
$$= \gamma(z_1 - z_2) - \tfrac{1}{2}\rho V_2^2 \qquad \textbf{(2)}$$

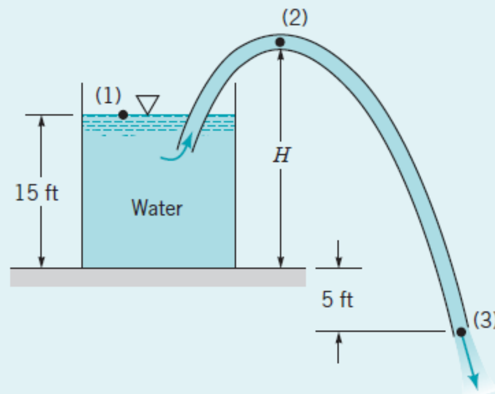
From Table B.1, the vapor pressure of water at 60 °F is 0.256 psia. Hence, for incipient cavitation the lowest pressure in the system will be $p = 0.256$ psia. Careful consideration of Eq. 2 and Fig. E3.10*b* will show that this lowest pressure will occur at the top of the hill. Since we have used gage pressure at point (1) ($p_1 = 0$), we must use gage pressure at point (2) also. Thus, $p_2 = 0.256 - 14.7 = -14.4$ psi and Eq. 2 gives

$$(-14.4 \text{ lb/in.}^2)(144 \text{ in.}^2/\text{ft}^2)$$
$$= (62.4 \text{ lb/ft}^3)(15 - H)\text{ft} - \tfrac{1}{2}(1.94 \text{ slugs/ft}^3)(35.9 \text{ ft/s})^2$$

or

$$H = 28.2 \text{ ft} \qquad \textbf{(Ans)}$$

For larger values of H, vapor bubbles will form at point (2) and the siphon action may stop.



EXAMPLE 3.8 Flow from a Tank—Pressure

GIVEN Air flows steadily from a tank, through a hose of diameter $D = 0.03$ m, and exits to the atmosphere from a nozzle of diameter $d = 0.01$ m as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure.

FIND Determine the flowrate and the pressure in the hose.

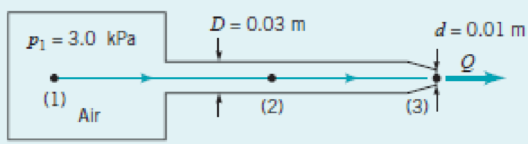


FIGURE E3.8

SOLUTION

If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet), this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2 \qquad \textbf{(1)}$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

$$\rho = \frac{p_1}{RT_1}$$
$$= [(3.0 + 101) \text{ kN/m}^2] \times \frac{10^3 \text{ N/kN}}{(286.9 \text{ N}\cdot\text{m/kg}\cdot\text{K})(15 + 273)\text{K}}$$
$$= 1.26 \text{ kg/m}^3$$

Thus, we find that

$$V_3 = \sqrt{\frac{2(3.0 \times 10^3 \text{ N/m}^2)}{1.26 \text{ kg/m}^3}} = 69.0 \text{ m/s}$$

or

$$Q = A_3V_3 = \frac{\pi}{4}d^2V_3 = \frac{\pi}{4}(0.01\text{ m})^2(69.0\text{ m/s})$$
$$= 0.00542\text{ m}^3/\text{s} \qquad \textbf{(Ans)}$$

The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 3.19)

$$A_2V_2 = A_3V_3$$

Hence,

$$V_2 = A_3V_3/A_2 = \left(\frac{d}{D}\right)^2 V_3$$
$$= \left(\frac{0.01\text{ m}}{0.03\text{ m}}\right)^2(69.0\text{ m/s}) = 7.67\text{ m/s}$$

and from Eq. 1

$$p_2 = 3.0 \times 10^3\text{ N/m}^2 - \tfrac{1}{2}(1.26\text{ kg/m}^3)(7.67\text{ m/s})^2$$
$$= (3000 - 37.1)\text{N/m}^2 = 2963\text{ N/m}^2 \qquad \textbf{(Ans)}$$

References

[1] Bansal, RK. ***A Textbook of Fluid Mechanics and Hydraulic Machines***: Laxmi publications, 2004.

[2] Y. A. ÇENGEL and J. M. CIMBALA, ***FLUID MECHANICS: FUNDAMENTALS AND APPLICATIONS***. New York: McGraw-Hill, 2006.

[3] RK Rajput. ***A Textbook of Fluid Mechanics and Hydraulic Machines***, 2008.

[4] Munson, Young, Okiishi's and Huebsch ***Fundamentals of Fluid Mechanics***, six edition.

10. Forces Acting on a Control Volume

Many different kinds of forces may act on a fluid element or a Control Volume, they may all be put into broad categories as described in Fig.6.

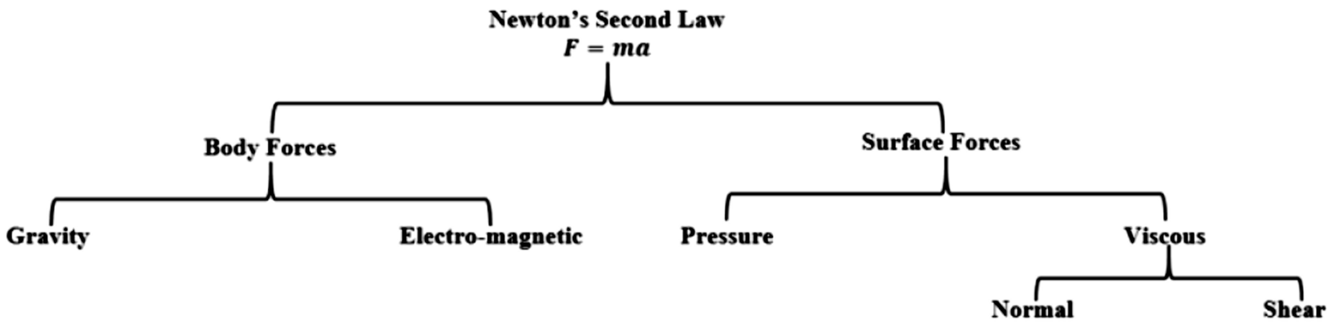


Fig.6. Forces acting on a Control Volume

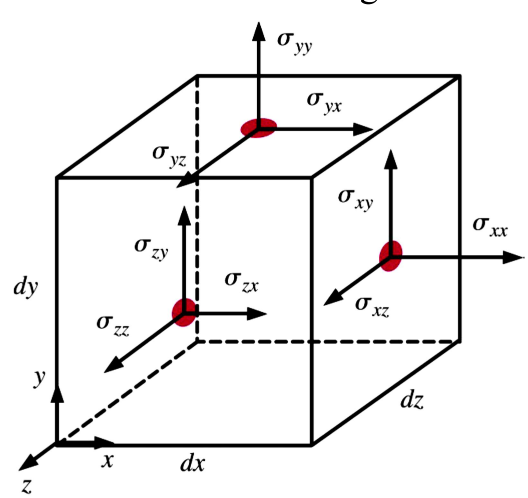
1) Body Forces: These are forces that act at a distance from the fluid element originate from a remote source, (act throughout the entire body of the CV) . Examples of these are gravitational, electrical and magnetic forces. The total body force acting on a fluid element is proportional to its mass. Total body force acting on CV:

$$\sum \vec{F}_{body} = \int_{CV} \rho \vec{g} d\mathcal{V} = m_{CV}\vec{g}$$

2) Surfaces Forces: These relate to forces that act by direct contact between the fluid element and the surrounding medium, (act on the control surface). Surface forces are as a result of only two sources:
   (i) The pressure forces due to pressure distribution on the surface, imposed by the outside fluid surrounding the fluid element.
   (ii) The viscous forces due to normal and shear stress distribution. Acting on the surface of the fluid element, imposed by the surrounding flow which tends to drag/push on the surface by means of friction.

Surface forces are not as simple to analyze since they include both normal and tangential components

- Diagonal components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ are called **normal stresses** and are due to pressure and viscous stresses.
- Off-diagonal components ($\sigma_{xy}, \sigma_{xz}, \sigma_{yx}, etc \ldots$) are called **shear stresses.**



Total surface force acting on Control Surface:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

Careful selection of CV allows expression of total force in terms of more readily available quantities like weight, pressure, and reaction forces.Goal is to choose CV to expose only the forces to be determined and a minimum number of other forces.

$$\sum \vec{F} = \underbrace{\sum \vec{F}_{gravity}}_{\text{body force}} + \underbrace{\sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other}}_{\text{surface forces}}$$

11. Linear momentum equation:

The momentum conservation equation (Linear momentum equation) is governed by Newton's second law of motion which states that the rate of change of momentum of a body is equal to the net force acting on it.

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)$$

Use Reynolds transport theorem with $b = \vec{V}$ and $B = m\,\vec{V}$ to shift from system formulation of the control volume formulation

$$\frac{d\left(m\vec{V}\right)_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$$

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$$

Steady flow	Uniform flow (V=V$_{avg}$)
$\sum \vec{F} = \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right) dA$	$\int_{CS} \rho\vec{V}\left(\vec{V} \cdot \vec{n}\right) dA \approx \rho V_{avg} A_c \vec{V}_{avg} = \dot{m}\vec{V}_{avg}$

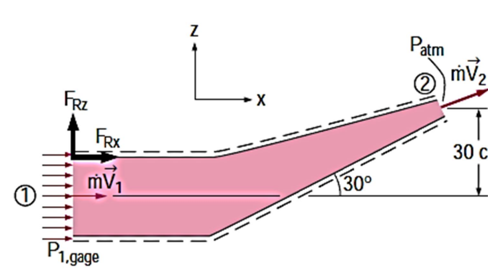
The algebraic form of **momentum equation** for a fixed control volume is then written as:

$$\sum \vec{F} = \frac{d}{dt}\int \rho\vec{V}\, d\mathcal{V} + \sum_{out} \beta\dot{m}\vec{V}_{avg} - \sum_{in} \beta\dot{m}\vec{V}_{avg}$$

Where β, called the **momentum-flux correction factor**

$$\beta = \frac{1}{A_c}\int_{A_C}\left(\frac{V}{V_{avg}}\right)^2 dA_c$$

Example (13): A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm^2 at the inlet and 7 cm^2 at the outlet. The elevation difference between the centers of the outlet and the inlet is 30 cm. The



weight of the elbow and the water in it is considered to be negligible. Determine (*a*) the gage pressure at the center of the inlet of the elbow and (*b*) the anchoring force needed to hold the elbow in place.

Solution: The continuity equation for this one-inlet, one-outlet, steady-flow system is m=14 kg/s. Noting that $m = \rho A V$, the inlet and outlet velocities of water are

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(0.0113 \text{ m}^2)} = 1.24 \text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(7 \times 10^{-4} \text{ m}^2)} = 20.0 \text{ m/s}$$

We use the Bernoulli equation (Chap. 5) as a first approximation to calculate the pressure. In Chap. 8 we will learn how to account for frictional losses along the walls. Taking the center of the inlet cross section as the reference level ($z_1 = 0$) and noting that $P_2 = P_{atm}$, the Bernoulli equation for a streamline going through the center of the elbow is expressed as

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_1 - P_{atm} = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2) \times \left(\frac{(20 \text{ m/s})^2 - (1.24 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0.3 - 0\right)\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right)$$

$$P_{1,\text{ gage}} = 202.2 \text{ kN/m}^2 = \mathbf{202.2 \text{ kPa}} \quad \text{(gage)}$$

(*b*) The momentum equation for steady one-dimensional flow is

$$\sum \vec{F} = \sum_{out} \beta \dot{m} \vec{V} - \sum_{in} \beta \dot{m} \vec{V}$$

We let the *x*- and *z*-components of the anchoring force of the elbow be F_{Rx} and F_{Rz}, and assume them to be in the positive direction. We also use gage pressure since the atmospheric pressure acts on the entire control surface. Then the momentum equations along the *x*- and *z*-axes become

$$F_{Rx} + P_{1,\text{ gage}}A_1 = \beta \dot{m} V_2 \cos\theta - \beta \dot{m} V_1$$

$$F_{Rz} = \beta \dot{m} V_2 \sin\theta$$

Solving for F_{Rx} and F_{Rz}, and substituting the given values,

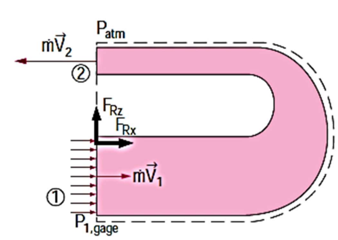
$$F_{Rx} = \beta \dot{m}(V_2 \cos\theta - V_1) - P_{1,\text{ gage}}A_1$$

$$= 1.03(14 \text{ kg/s})[(20 \cos 30^\circ - 1.24) \text{ m/s}]\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) - (202{,}200 \text{ N/m}^2)(0.0113 \text{ m}^2)$$

$$= 232 - 2285 = \mathbf{-2053 \text{ N}}$$

$$F_{Rz} = \beta \dot{m} V_2 \sin\theta = (1.03)(14 \text{ kg/s})(20 \sin 30^\circ \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right) = \mathbf{144 \text{ N}}$$

Example (14): The deflector elbow in Example 13 is replaced by a reversing elbow such that the fluid makes a 180° U-turn before it is discharged, as shown in Fig. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place.



SOLUTION The inlet and the outlet velocities and the pressure at the inlet of the elbow remain the same, but the vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case ($F_{Rz} = 0$) since there is no other force or momentum flux in the vertical direction (we are neglecting the weight of the elbow and the water). The horizontal component of the anchoring force is determined from the momentum equation written in the *x*-direction. Noting that the outlet velocity is negative since it is in the negative *x*-direction, we have

$$F_{Rx} + P_{1,\,gage}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

Solving for F_{Rx} and substituting the known values,

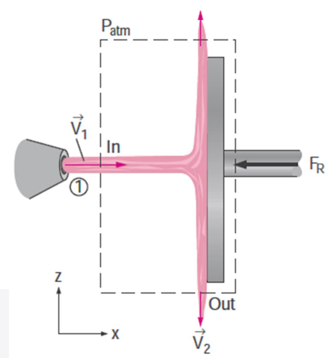
$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\,gage}A_1$$

$$= -(1.03)(14\ \text{kg/s})[(20 + 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2)$$

$$= -306 - 2285 = \mathbf{-2591\ N}$$

Therefore, the horizontal force on the flange is 2591 N acting in the negative *x*-direction (the elbow is trying to separate from the pipe). This force is equivalent to the weight of about 260 kg mass, and thus the connectors (such as bolts) used must be strong enough to withstand this force.

Example (15): Water is accelerated by a nozzle to an average speed of 20 m/s, and strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s . After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream.



Analysis We draw the control volume for this problem such that it contains the entire plate and cuts through the water jet and the support bar normally. The momentum equation for steady one-dimensional flow is given as

$$\sum \vec{F} = \sum_{out} \beta\dot{m}\vec{V} - \sum_{in} \beta\dot{m}\vec{V}$$

Writing it for this problem along the *x*-direction (without forgetting the negative sign for forces and velocities in the negative *x*-direction) and noting that $V_{1,\,x} = V_1$ and $V_{2,\,x} = 0$ gives

$$-F_R = 0 - \beta\dot{m}\vec{V}_1$$

Substituting the given values,

$$F_R = \beta\dot{m}\vec{V}_1 = (1)(10\ \text{kg/s})(20\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{200\ N}$$

Power Mechanics Technical

Engineering Department

Second Stage

Fluid Mechanics - Dynamic

Chapter Three: Experimental Modeling

M.Sc Zahraa A. Nadeem

1 . Introduction

Many practical engineering problems involving fluid mechanics can be solved experimentally, Few problems in fluids can be solved by analysis alone.

Correlate experiments to a specific problem, the goal is to make the experiment widely applicable.Similitude is used to make experiments more applicable, Laboratory flows are studied under carefully controlled conditions.

2 . Dimensions And Units

A **dimension** is a measure of a physical quantity (without numerical values), while a **unit** is a way to assign a *number* to that dimension.

There are seven **primary dimensions** (also called **fundamental** or **basic dimensions**) *mass, length, time, temperature, electric current, amount of light, and amount of matter.*

All nonprimary dimensions can be formed by some combination of the seven primary dimensions.

Primary dimensions and their associated primary SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time†	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

For example, force

$$\text{Dimensions of force:} \quad \{\text{Force}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\right\} = \{\text{mL/t}^2\}$$

3 . Dimensional Analysis

Dimensional analysis *is a mathematical technique which makes use of the study of the dimensions for solving several engineering problems.* Each physical phenomenon can be expressed by an equation giving relationship between different quantities, such quantities are dimensional and non-dimensional.

An important characteristic of this system, which would be of interest to an engineer designing a pipeline, is the pressure drop per unit length that develops along the pipe as a result of friction. It cannot generally be solved analytically without the use of experimental data.

First, we determine the important variables in the flow related to pressure drop, We expect the list to include the pipe diameter, D, the fluid density (ρ), fluid viscosity(μ), and the mean velocity (V), at which the fluid is flowing through the pipe. Thus, we can express this relationship as

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

So, how do we approach this problem?

Logically, it seems that we could vary one variable at a time holding the other constants fig.(1).

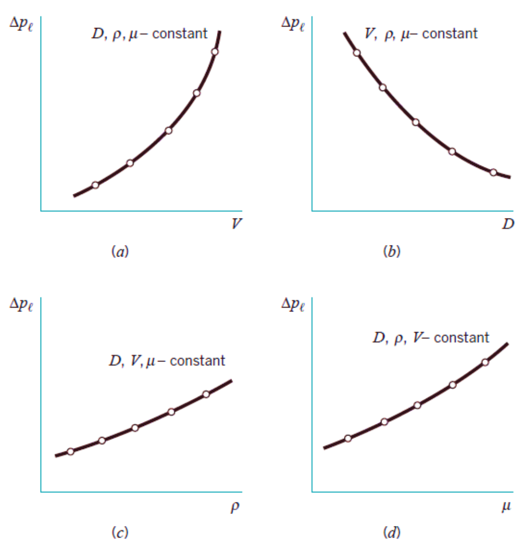


Fig.(1) How the pressure drop in a pipe may be affected by several different factors.

We can collect these variables into two nondimensional combinations of variables (called ***dimensionless products*** or *dimensionless groups*) so that

$$\frac{D\,\Delta p_\ell}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Now instead of working with 5 variables, there are only two.

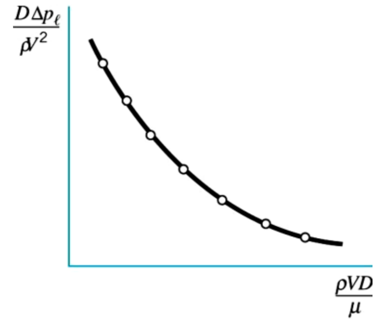


Fig.(2) plot of pressure drop data using dimensionless parameters.

Then, we check our dimensionless groups

$$V \doteq LT^{-1},\ \Delta p_\ell \doteq FL^{-3}, D \doteq L, \rho \doteq FL^{-4}T^2, \mu \doteq FL^{-2}T,$$

Substituting, we see no dimensions on our two variables:

$$\frac{D\,\Delta p_\ell}{\rho V^2} \doteq \frac{L(F/L^3)}{(FL^{-4}T^2)(LT^{-1})^2} \doteq F^0L^0T^0$$

$$\frac{\rho VD}{\mu} \doteq \frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)} \doteq F^0L^0T^0$$

Not only have we reduced the number of variables from five to two, but the new groups are dimensionless combinations of variables, which means that the results will be independent of the system of units we choose to use. This type of analysis is called *dimensional analysis*, and the basis for its application to a wide variety of problems is found in the *Buckingham pi theorem* described in the following section.

4 . Buckingham Pi Theorem

Buckingham Pi Theorem is a systematic way of forming dimensionless groups:

If an equation involving k variables is dimensionally homogeneous, it can be reduced to a relationship among $k - r$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables.

The dimensionless products are referred to as "pi terms". The pi theorem is based on the idea of dimensional homogeneity

$$u_1 = f(u_2, u_3, \ldots, u_k)$$

Dimensions on the left side = dimension on the right side. It then follows that we can rearrange the equation into a set of dimensionless products (pi terms) so that

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

The required number of pi terms is fewer than the original number of variables by r, where r is the minimum number of reference dimensions needed to describe the original set of variables (M, L, T, or F).

5 . Determination of Pi Terms

To illustrate various steps we will consider the steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal circular pipe. We are interested in the pressure drop.

Step 1. List all the variables that are involved in the problem:

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Step 2. Express each of the variables in terms of basic dimensions:

$$\Delta p_\ell \doteq FL^{-3}$$
$$D \doteq L$$
$$\rho \doteq FL^{-4}T^2$$
$$\mu \doteq FL^{-2}T$$
$$V \doteq LT^{-1}$$

Step 3. Determine the require number of pi terms:

The basic dimensions are F,L,T or M,L,T, noting $F = MLT^{-2}$, 3 total Then the number of pi terms are the number of variables, 5 minus the number of basic dimensions, 3. So there should be **two pi terms** for this case.

Step 4. Select a number of repeating variables, where the number required is equal to the number of reference dimensions. We choose three independent variables as the repeating variables—there can be more than one set of repeating variables.

Repeating variables: **D, V, and ρ**

We note the these three variables by themselves are dimensionally independent; you can not form a dimensionless group with them alone.

Step 5. Form a pi term by multiplying one of the nonrepeating variables by the product of repeating variables each raised to an exponent that will make the

combination dimensionless. The first group chosen usually includes the dependent variable.

$$\Pi_1 = \Delta p_\ell D^a V^b \rho^c$$

Product should be dimensionless:

$$(FL^{-3})(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

So, we need to solve for the exponent values.

$$\begin{aligned} 1 + c &= 0 \qquad &(\text{for } F) \\ -3 + a + b - 4c &= 0 \qquad &(\text{for } L) \\ -b + 2c &= 0 \qquad &(\text{for } T) \end{aligned}$$

Solving the set of algebraic equations, we obtain: a = 1, b = -2, c = -1:

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2}$$

μ is a remaining nonrepeating variable, so we can form another group:

$$\Pi_2 = \mu D^a V^b \rho^c$$

$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

Solving, a = -1, b = -1, and c = -1

$$\Pi_2 = \frac{\mu}{DV\rho}$$

Step 6. Repeat Step 5. for each of the remaining repeating variables.

Step 7. Check all the resulting pi terms to make sure they are dimensionless.

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq \frac{(FL^{-3})(L)}{(FL^{-4}T^2)(LT^{-1})^2} \doteq F^0L^0T^0$$

$$\Pi_2 = \frac{\mu}{DV\rho} \doteq \frac{(FL^{-2}T)}{(L)(LT^{-1})(FL^{-4}T^2)} \doteq F^0L^0T^0$$

Step 8. Express the final form as relationship among the pi terms and think about what it means.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

For our case,

$$\frac{\Delta p_\ell D}{\rho V^2} = \tilde{\phi}\left(\frac{\mu}{DV\rho}\right) \quad \text{or} \quad \frac{D\ \Delta p_\ell}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right)$$

Pressure drop depends on the Reynolds Number.

6 . Selection of Variables

One of the most important aspects of dimensional analysis is choosing the variables important to the flow, however, this can also prove difficult. We do not want to choose so many variables that the problem becomes cumbersome.

Often we use engineering simplifications, to obtain first order results sacrificing some accuracy, but making the study more tangible. Most variables fall in to the categories of geometry, material property, and external effects:

- Geometry: lengths and angles, usually very important and obvious variables.
- Material Properties: bind the relationship between external effects and the fluid response. Viscosity, and density of the fluid.
- External Effects: Denotes a variable that produces a change in the system, pressures, velocity, or gravity.

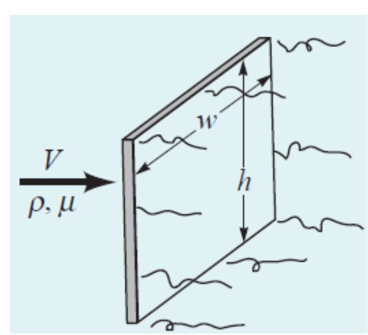
We must choose the variables such that they are independent:

$$f(p, q, r, \ldots, u, v, w, \ldots) = 0$$

$$q = f_1(u, v, w, \ldots)$$

Then, u, v, and w are not necessary in f if they only enter the problem through q, or q is not necessary in f.

Example (1): A thin rectangular plate having a width *w* and a height *h* is located so that it is normal to a moving stream of fluid as shown in Fig. Assume the drag, D, that the fluid exerts on the plate is a function of *w* and *h*, the fluid viscosity and density, ρ and μ, respectively, and the velocity *V* of the fluid approaching the plate. Determine a suitable set of pi terms to study this problem experimentally.



Solution

From the statement of the problem we can write

$$\mathscr{D} = f(w, h, \mu, \rho, V)$$

where this equation expresses the general functional relationship between the drag and the several variables that will affect it. The dimensions of the variables (using the *MLT* system) are

$$\mathscr{D} \doteq MLT^{-2}$$
$$w \doteq L$$
$$h \doteq L$$
$$\mu \doteq ML^{-1}T^{-1}$$
$$\rho \doteq ML^{-3}$$
$$V \doteq LT^{-1}$$

We see that all three basic dimensions are required to define the six variables so that the Buckingham pi theorem tells us that three pi terms will be needed (six variables minus three reference dimensions, $k - r = 6 - 3$).

We will next select three repeating variables such as w, V, and ρ. A quick inspection of these three reveals that they are dimensionally independent, since each one contains a basic dimension not included in the others. Note that it would be incorrect to use both w and h as repeating variables since they have the same dimensions.

Starting with the dependent variable, $\mathscr{D}$, the first pi term can be formed by combining $\mathscr{D}$ with the repeating variables such that

$$\Pi_1 = \mathscr{D} w^a V^b \rho^c$$

and in terms of dimensions

$$(MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

Thus, for Π_1 to be dimensionless it follows that

$$1 + c = 0 \qquad (\text{for } M)$$
$$1 + a + b - 3c = 0 \qquad (\text{for } L)$$
$$-2 - b = 0 \qquad (\text{for } T)$$

and, therefore, $a = -2$, $b = -2$, and $c = -1$. The pi term then becomes

$$\Pi_1 = \frac{\mathscr{D}}{w^2V^2\rho}$$

Next the procedure is repeated with the second nonrepeating variable, h, so that

$$\Pi_2 = h w^a V^b \rho^c$$

It follows that

$$(L)(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

and

$$c = 0 \qquad (\text{for } M)$$
$$1 + a + b - 3c = 0 \qquad (\text{for } L)$$
$$b = 0 \qquad (\text{for } T)$$

so that $a = -1$, $b = 0$, $c = 0$, and therefore

$$\Pi_2 = \frac{h}{w}$$

The remaining nonrepeating variable is μ so that

$$\Pi_3 = \mu w^a V^b \rho^c$$

with

$$(ML^{-1}T^{-1})(L)^a(LT^{-1})^b(ML^{-3})^c \doteq M^0L^0T^0$$

and, therefore,

$$1 + c = 0 \qquad (\text{for } M)$$
$$-1 + a + b - 3c = 0 \qquad (\text{for } L)$$
$$-1 - b = 0 \qquad (\text{for } T)$$

Solving for the exponents, we obtain $a = -1$, $b = -1$, $c = -1$ so that

$$\Pi_3 = \frac{\mu}{wV\rho}$$

Now that we have the three required pi terms we should check to make sure they are dimensionless. To make this check we use F, L, and T, which will also verify the correctness of the original dimensions used for the variables. Thus,

$$\Pi_1 = \frac{\mathscr{D}}{w^2V^2\rho} \doteq \frac{(F)}{(L)^2(LT^{-1})^2(FL^{-4}T^2)} \doteq F^0L^0T^0$$

$$\Pi_2 = \frac{h}{w} \doteq \frac{(L)}{(L)} \doteq F^0L^0T^0$$

$$\Pi_3 = \frac{\mu}{wV\rho} \doteq \frac{(FL^{-2}T)}{(L)(LT^{-1})(FL^{-4}T^2)} \doteq F^0L^0T^0$$

example, we could express the final result in the form

$$\frac{\mathscr{D}}{w^2\rho V^2} = \phi\left(\frac{w}{h}, \frac{\rho V w}{\mu}\right) \qquad \textbf{(Ans)}$$

which would be more conventional, since the ratio of the plate width to height, w/h, is called the *aspect ratio*, and $\rho Vw/\mu$ is the Reynolds number.

7 . Dimensionless Groups in Fluid Mechanics

Some Common Variables and Dimensionless Groups in Fluid Mechanics

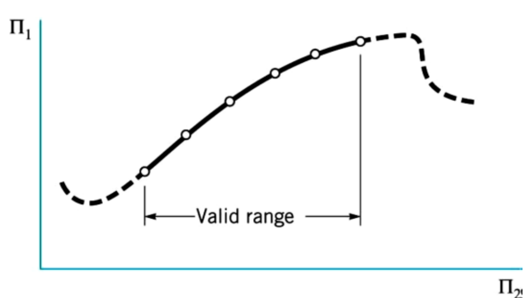
Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{V}{c}$	Mach number,[a] Ma	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\frac{\omega \ell}{V}$	Strouhal number, St	$\frac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\frac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

- If only one pi variable exists in a fluid phenomenon, the functional relationship must be a constant. The constant must be determine from experiment.

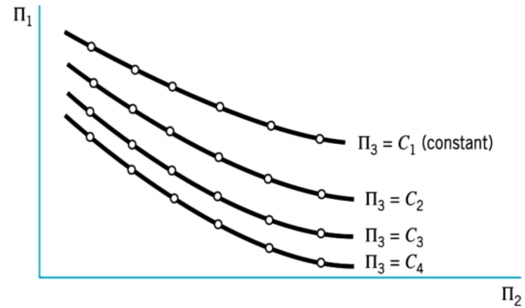
$$\Pi_1 = C$$

- If we have two pi terms, we must be careful not to over extend the range of applicability, but the relationship can be presented pretty easily graphically:



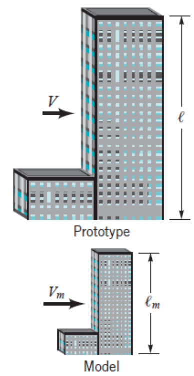
$$\Pi_1 = \phi(\Pi_2)$$

- If we have three pi groups, we can represent the data as a series of curves, however, as the number of pi terms increase the problem becomes less tractable, and we may resort to modeling specific characteristics.



8 . Modeling and Similitude

*A **model** is a representation of a physical system that may be used to predict the behavior of the system in some desired respect*. The physical system for which the predictions are to be made is called the ***prototype.***



Geometric similarity: A model and a prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear scale ratio. (All angles are preserved,all flow directions are the same, and orientations must be the same).

$$\frac{\ell_1}{\ell_2} = \frac{\ell_{1m}}{\ell_{2m}}$$

Kinematic Similarity: Same length scale ratio and same time-scale ratio. The motion of the system is kinematically similar if homolgous particles lie at homolgous locations at homologous times. This requires equivalence of dimensionless groups:

Reynolds Number, Froude Number, Mach numbers, etc.

Length scale:

Froude Number similarity: $\dfrac{V_m}{\sqrt{g_m \ell_m}} = \dfrac{V}{\sqrt{g\ell}}$ $\qquad \dfrac{V_m}{V} = \sqrt{\dfrac{\ell_m}{\ell}} = \sqrt{\lambda_\ell}$

Reynolds Number similarity: $\dfrac{\rho_m V_m \ell_m}{\mu_m} = \dfrac{\rho V \ell}{\mu}$ $\qquad \dfrac{V_m}{V} = \dfrac{\mu_m}{\mu}\dfrac{\rho}{\rho_m}\dfrac{\ell}{\ell_m}$

Then, $\dfrac{\mu_m/\rho_m}{\mu/\rho} = \dfrac{\nu_m}{\nu} = (\lambda_\ell)^{3/2}$ Might relax condition.

Time scale: $\dfrac{t_m}{t} = \dfrac{l_m/v_m}{l/v} = \sqrt{\lambda_l}$

Dynamic Similarity: the same length scale, time-scale, and force scale is required. First, satisfy geometric, and kinematic similarity. Dynamic similarity then exists if the force and pressure coefficient are the same.

In order to ensure that the force and pressure coefficients are the same:

- For compressible flow: ***Re, Mach, and specific heat ratio*** must be matched.
- For incompressible flow with no free surface: ***Re*** matching only.
- For incompressible flow with a free surface: ***Re, Froude***, and possibly ***Weber number*** (surface tension effects), and ***cavitation number*** must be matched.

Example (2): The excess pressure inside a bubble is known to be dependent on bubble radius and surface tension. After finding the pi terms, determine the variation in excess pressure if we **(a)** double the radius and **(b)** double the surface tension.

Given $\Delta p = f(R, \sigma)$, where $\Delta p \doteq \frac{F}{L^2} = \frac{M}{LT^2}$, $R \doteq L$, and $\sigma \doteq \frac{F}{L} = \frac{M}{T^2}$

Consider the (MLT) units so that

$k - r = 3 - 3 = 0$ since there 3 variables and 3 dimensions.

According to this, there should be $k-r=0$ pi terms!?

However, if we consider the (FLT) units we see that it takes only F and L, T is not needed, so that $r=2$.

Hence, $k-r = 3-2 = 1$, so only 1 pi term is needed.

That is, $\Pi_1 = \text{constant}$

To determine Π_1, consider

$\Pi_1 = \Delta p \, R^a \sigma^b$ or

$\Delta p \, R^a \sigma^b \doteq \frac{F}{L^2} L^a \left(\frac{F}{L}\right)^b = F^{1+b} \, L^{-2+a-b}$

Thus:

$F: \quad 1 + b = 0$

$L: \quad a - b - 2 = 0$

or $b = -1$ and $a = b + 2 = 1$

Hence $\underline{\underline{\Pi_1 = \frac{\Delta p R}{\sigma}}}$ or $\frac{\Delta p R}{\sigma} = C$, where $C = \text{constant}$.

or

$$\Delta p = \frac{C\sigma}{R} \qquad (1)$$

(a) If R is doubled, $\underline{\underline{\Delta p \text{ is reduced by half.}}}$ (See Eq. (1))

(b) If σ is doubled, $\underline{\underline{\Delta p \text{ is doubled.}}}$ (See Eq. (1))

Example (3): The pressure rise, Δp, across a pump can be expressed as

$$\Delta p = f(D, \rho, \omega, Q)$$

where D is the impeller diameter, ρ the fluid density, ω the rotational speed, and Q the flowrate. Determine a suitable set of dimensionless parameters.

$$\Delta p \doteq FL^{-2} \quad D \doteq L \quad \rho \doteq FL^{-4}T^{2} \quad \omega \doteq T^{-1} \quad Q \doteq L^{3}T^{-1}$$

From the pi theorem, $5-3=2$ pi terms required. Use D, ρ, and ω as repeating variables. Thus,

$$\pi_1 = \Delta p\, D^a \rho^b \omega^c$$

and so that

$$(FL^{-2})(L)^a(FL^{-4}T^{2})^b(T^{-1})^c \doteq F^0L^0T^0$$

$$1+b=0 \qquad \text{(for } F\text{)}$$
$$-2+a-4b=0 \qquad \text{(for } L\text{)}$$
$$2b-c=0 \qquad \text{(for } T\text{)}$$

It follows that $a=-2$, $b=-1$, $c=-2$, and therefore

$$\pi_1 = \frac{\Delta p}{D^2 \rho \omega^2}$$

Check dimensions using MLT system:

$$\frac{\Delta p}{D^2 \rho \omega^2} \doteq \frac{ML^{-1}T^{-2}}{(L)^2(ML^{-3})(T^{-1})^2} \doteq M^0L^0T^0 \qquad \therefore \text{ok}$$

For π_2:

$$\pi_2 = Q\, D^a \rho^b \omega^c$$

$$(L^3T^{-1})(L)^a(FL^{-4}T^{2})^b(T^{-1})^c \doteq F^0L^0T^0$$

$$b=0 \qquad \text{(for } F\text{)}$$
$$3+a-4b=0 \qquad \text{(for } L\text{)}$$
$$-1+2b-c=0 \qquad \text{(for } T\text{)}$$

It follows that $a=-3$, $b=0$, $c=-1$, and therefore

$$\pi_2 = \frac{Q}{D^3 \omega}$$

Check dimensions using MLT system:

$$\frac{Q}{D^3 \omega} \doteq \frac{L^3T^{-1}}{(L)^3(T^{-1})} \doteq M^0L^0T^0 \qquad \therefore \text{ok}$$

Thus,

$$\underline{\underline{\frac{\Delta p}{D^2 \rho \omega^2} = \phi\left(\frac{Q}{D^3 \omega}\right)}}$$